Proof of alternate form of second shifting theorem

Suppose $L\{F(t)\} = f(s)$ and $c > 0$.

Also $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

To prove that

$$L\{F(t-c).H(t-c)\} = e^{-cs}.f(s)$$

Now, $L\{F(t-c).H(t-c)\} = \int_0^\infty e^{-st} F(t-c)\, H(t-c)\, dt.$

Let $t - c = x.$

$dt = dx$

Also when $t = 0$ $x = -c$

& when $t = \infty$ $x = \infty$

$$= \int_{-c}^{\infty} e^{-s(c+x)}. F(x). H(x)\, dx$$

$$= \int_{-c}^{0} e^{-s(c+x)}. F(x). H(x)dx + \int_0^\infty e^{-s(c+x)}. F(x) H(x)\, dx$$

$$= \int_{-c}^{0} e^{-s(c+x)}. F(x). 0. dx + \int_0^\infty e^{-s(c+x)}. F(x). 1\, dx$$

$$= 0 + e^{-sc} \int_0^\infty e^{-sx}. F(x)\, dx$$

$$= e^{-sc}. \int_0^\infty e^{-st}. F(t)\, .dt$$

$$= e^{-sc}. f(s)$$ Proved

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$\mathcal{T}^{n}$: <u>Change of scale</u>: If $L\{F(t)\} = f(s)$ then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

<u>Proof</u>. We have

$$L\{F(at)\} = \int_0^\infty e^{-st} \cdot F(at) \, dt$$

Let $at = u$

$$\Rightarrow a\,dt = du \Rightarrow dt = \frac{du}{a}$$

Also when $t=0$ $u=0$ & $t=\infty$ $u=\infty$

$$= \int_0^\infty e^{-su/a} \cdot F(u) \cdot \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{su}{a}} F(u) \, du$$

$$= \frac{1}{a} \int_0^\infty e^{-pu} \cdot F(u) \, du \quad \left(\text{taking } \frac{s}{a} = p\right)$$

$$= \frac{1}{a} f(p)$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right).$$

Proved

<u>Some standard results</u>:

(1) $L\{1\} = \frac{1}{s}$.

<u>Proof</u> $L\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1)$$

$$= 1/s.$$

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② $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof $L\{t^n\} = \int_0^\infty e^{-st} t^n . dt$

Let $st = x$ ie $t = \frac{x}{s}$

$\Rightarrow s dt = dx$ $dt = \frac{dx}{s}$

When $t=0$, $x=0$ & When $t=\infty$ $x=\infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

$$= \frac{1}{s^{n+1}} \cdot \Gamma(n+1)$$

{Note: If n is a positive integer then $\Gamma(n+1) = \underline{|n}$}

In this case $L\{t^n\} = \frac{\underline{|n}}{s^{n+1}}$

③ $L\{e^{at}\} = \frac{1}{s-a}$ if $s > a$.

Proof We have

$$L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at} . dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

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4) $L\{\cos at\} = \dfrac{s}{s^2+a^2}$ & $L\{\sin at\} = \dfrac{a}{s^2+a^2}$.

Proof We know that

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$\Rightarrow L\{e^{iat}\} = \frac{1}{(s-ia)}$$

$$= \frac{s+ia}{(s-ia)(s+ia)}.$$

$$= \frac{s+ia}{s^2+a^2}$$

$$\Rightarrow L\{\cos at + i \sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Using linear property-

$$L\{\cos at\} + i\,L\{\sin at\} = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Equating real parts

$$L\{\cos at\} = \frac{s}{s^2+a^2}.$$

Equating imaginary parts.

$$L\{\sin at\} = \frac{a}{s^2+a^2}.$$

5) $L\{e^{at}t^n\} = \dfrac{\lfloor n}{(s-a)^{n+1}}$ if n is +ve integer & $s>a$

Proof $L\{t^n\} = \dfrac{\lfloor n}{s^{n+1}}$ —— (1)

Also we know that if $L\{F(t)\} = f(s)$ then

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$$L\{e^{at}F(t)\} = f(s-a).$$

$$\Rightarrow L\{e^{at}\cdot t^n\} = \frac{\lfloor n}{(s-a)^{n+1}}$$

Proved

(6) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$ $s > |a|$

& $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$ $s > |a|$

Proof.

$$L\{\sinh at\} = L\left\{\frac{1}{2}(e^{at} - e^{-at})\right\}$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s}+a-\cancel{s}+a}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{\cancel{2}}\frac{\cancel{2}a}{(s-a)(s+a)}$$

$$= \frac{a}{s^2-a^2}$$

$$L\{\cosh at\} = L\left\{\frac{1}{2}(e^{at}+e^{-at})\right\}$$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{s}{s^2-a^2}$$

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