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TENSOR CALCULUS II (15)

Riemanian christoffel's tensor

Definition: christoffel symbols or Brackets:

We define $\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$ —— (1)

$$\Gamma^{\sigma}_{\mu\nu} = g^{\sigma\beta}\Gamma_{\mu\nu,\beta} \quad \text{—— (2)}$$

The first one ie $\Gamma_{\mu\nu,\sigma}$ is called christoffel symbol of the first kind, it is also written as $[\mu\nu,\sigma]$ and the second one ie $\Gamma^{\sigma}_{\mu\nu}$ is called christoffel symbol of the second kind, it is also written as $\left\{\begin{matrix}\sigma\\ \mu\nu\end{matrix}\right\}$.

Theorem (1) Prove that christoffel symbols $[JK,i]$ and $\left\{\begin{matrix}i\\ JK\end{matrix}\right\}$ are symmetric in J and K.

Proof: If we show that $[JK,i] = [KJ,i]$ and $\left\{\begin{matrix}i\\ JK\end{matrix}\right\} = \left\{\begin{matrix}i\\ KJ\end{matrix}\right\}$, then the result will follow.

Now $[JK,i] = \frac{1}{2}\left(\frac{\partial g_{Ki}}{\partial x^{J}} + \frac{\partial g_{Ji}}{\partial x^{k}} - \frac{\partial g_{JK}}{\partial x^{i}}\right)$ —— (1)

Interchanging J and K, we have

$$[KJ,i] = \frac{1}{2}\left(\frac{\partial g_{Ji}}{\partial x^{k}} + \frac{\partial g_{Ki}}{\partial x^{J}} - \frac{\partial g_{KJ}}{\partial x^{i}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{Ki}}{\partial x^{J}} + \frac{\partial g_{Ji}}{\partial x^{k}} - \frac{\partial g_{JK}}{\partial x^{i}}\right)$$

$$= [JK,i] \quad \text{by (1)}$$

for g_{ij} is a symmetric tensor

$\therefore \quad [jk, i] = [kj, i]$. ——— (2)

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further $\left\{ {i \atop jk} \right\} = g^{ih}[jk, h]$

$= g^{ih}[kj, h]$ by (2)

$= \left\{ {i \atop kj} \right\}$

$\therefore \left\{ {i \atop jk} \right\} = \left\{ {i \atop kj} \right\}$ ——— (3)

From (2) and (3), the required result follow.

Theorem (2) Prove that

(i) $\Gamma_{ij,k} + \Gamma_{jk,i} = \dfrac{\partial g_{ik}}{\partial x^j}$

(ii) $\Gamma^i_{ij} = \dfrac{\partial}{\partial x^j} \log \sqrt{-g}$

(iii) $\Gamma^i_{ij} = \dfrac{\partial}{\partial x^j} \log \sqrt{g}$

(iv) $\dfrac{\partial g^{ij}}{\partial x^k} = -g^{li}\Gamma^i_{lk} - g^{lj}\Gamma^j_{lk}$

Proof (i) By definition

$$\Gamma_{ij,k} + \Gamma_{jk,i} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ki}}{\partial x^j}\right) = \frac{\partial g_{ik}}{\partial x^j},$$

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Proof (ii) we know that

$$\frac{\partial a}{\partial x} = A^i_j \frac{\partial a^j_i}{\partial x} \quad \text{(in usual notation)}$$

In our case it becomes

$$\frac{\partial g}{\partial x^j} = (\text{Cofactor of } g_{ik}) \frac{\partial g_{ik}}{\partial x^j}$$

$$= g\, g^{ik} \frac{\partial g_{ik}}{\partial x^j}, \text{ for } g^{ik} = \frac{\text{Cofactor of } g_{ik}}{g}$$

$$\Rightarrow \frac{1}{g}\frac{\partial g}{\partial x^j} = g^{ik}\frac{\partial g_{ik}}{\partial x^j} = g^{ik}\left[\Gamma_{ij,k} + \Gamma_{jk,i}\right] \text{ by (i)}$$

$$= g^{ik}\Gamma_{ij,k} + g^{ik}\Gamma_{jk,i}$$

$$= \Gamma^i_{ij} + \Gamma^k_{jk} = \Gamma^i_{ij} + \Gamma^i_{ji} = 2\Gamma^i_{ij}$$

$$\Rightarrow \frac{1}{2g}\frac{\partial g}{\partial x^j} = \Gamma^i_{ij} \quad \text{——— (1)}$$

$$\Rightarrow \Gamma^i_{ij} = \frac{1}{2g}\frac{\partial g}{\partial x^j} = \frac{1}{2}\frac{\partial \log \sqrt{g}}{\partial x^j} \quad \text{——— (2)}$$

Proved

Also $$\frac{1}{2g}\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j}\log\sqrt{-g} \quad \text{——— (3)}$$

By (1) & (3), we have

$$\frac{1}{2g}\frac{\partial g}{\partial x^j} = \Gamma^i_{ij} \quad \text{——— (4)}$$

result (iii) is proved.

Proof (iv) $g_{ij}\, g^{jk} = \delta^k_i = 1 \text{ or } 0$

Differentiating w.r.t. x^m, we have

$$g^{jk}\frac{\partial g_{ij}}{\partial x^m} + g_{ij}\frac{\partial g^{jk}}{\partial x^m} = 0$$

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Multiplying both sides by g^{li}, and noting that $g_{ij} g^{li} = \delta^l_j$, $\delta^l_j \frac{\partial g^{jk}}{\partial x^m} = \frac{\partial g^{lk}}{\partial x^m}$

Now $g^{li} g^{jk} \frac{\partial g_{ij}}{\partial x^m} + g^{li} g_{ij} \frac{\partial g^{jk}}{\partial x^m} = 0$

$$\Rightarrow g^{li} g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \delta^l_j \frac{\partial g^{jk}}{\partial x^m} = 0$$

$$\Rightarrow g^{li} g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{lk}}{\partial x^m} = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} g^{jk} \left[\Gamma_{im,j} + \Gamma_{jm,i} \right] = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} g^{jk} \Gamma_{im,j} + g^{jk} g^{li} \Gamma_{jm,i} = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} \Gamma^k_{im} + g^{jk} \Gamma^l_{jm} = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{lk} \Gamma^k_{km} + g^{hk} \Gamma^l_{km} = 0$$

In view of this, we have

$$\Rightarrow \frac{\partial g^{ij}}{\partial x^k} + g^{li} \Gamma^i_{lk} + g^{lj} \Gamma^j_{lk} = 0$$

$$\Rightarrow \frac{\partial g^{ij}}{\partial x^k} = -g^{li} \Gamma^i_{ik} - g^{lj} \Gamma^j_{lk} = 0$$

Proved