Laplace Transform of Derivatives:

If $L\{F(t)\} = f(s)$, then

$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - sF^{n-2}(0) - F^{n-1}(0)$$

where $F^n(t) = \dfrac{d^n F(t)}{dt^n}$

Proof Given $L\{F(t)\} = f(s)$.

$$L\{F'(t)\} = \int_0^\infty e^{-st} \cdot \frac{dF(t)}{dt} \cdot dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty - \int_0^\infty -s \cdot e^{-st} \cdot F(t)\, dt$$

$$= \{0 \cdot \cancel{F(\infty)} - F(0)\} + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= -F(0) + sf(s)$$

$$= sf(s) - F(0) \quad \text{——— (1)}$$

Let $G(t) = F'(t)$ & $L\{G(t)\} = g(s)$

$\Rightarrow L\{G'(t)\} = sg(s) - G(0)$

$\Rightarrow L\{G''(t)\} = s\, L\{G(t)\} - G(0)$

$\Rightarrow L\{F''(t)\} = s\, L\{F'(t)\} - F'(0)$

$\Rightarrow L\{F''(t)\} = s\{sf(s) - F(0)\} - F'(0)$

$\Rightarrow L\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$ ——— (2)

Generalising (1) & (2)

$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - sF^{n-2}(0) - F^{n-1}(0)$$

Ans

Laplace Transform of integral:

If $L\{F(t)\} = f(s)$ then $L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s)$.

Proof

Given $L\{F(t)\} = f(s)$.

Let $G(t) = \int_0^t F(u)\,du$ ——— ①

Clearly $G(0) = \int_0^0 F(u)\,du = 0$

$$G'(t) = \frac{d}{dt}\left\{\int_0^t F(u)\,du\right\} = F(t)$$

We know that

$$L\{G'(t)\} = s.L\{G(t)\} - G(0)$$

$$\Rightarrow L\{F(t)\} = s.L\left\{\int_0^t F(u)\,du\right\} - 0$$

$$\Rightarrow f(s) = s\,L\left\{\int_0^t F(u)\,du\right\}$$

$$\Rightarrow L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s)$$

Multiplication with power of t

If $L\{F(t)\} = f(s)$ then $L\{t^n F(t)\} = (-1)^n \cdot \frac{d^n}{ds^n} f(s)$

For $n = 1, 2, 3, \ldots$

Proof Let $L\{F(t)\} = f(s)$. and $\frac{d^n}{ds^n} f(s) = f^{(n)}(s)$.

Now $f(s) = \int_0^\infty e^{-st} \cdot F(t)\,dt$

Differentiating both sides w.r.to s

$$\frac{d}{ds}\left(f(s)\right) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\,dt$$

$$\Rightarrow \frac{df(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) F(t)\,dt.$$

$$\Rightarrow \frac{df(s)}{ds} = \int_0^\infty -t\, e^{-st} F(t)\,dt$$

$$\Rightarrow (-1)^1 \frac{df(s)}{ds} = \int_0^\infty t\, e^{-st} F(t)\,dt.$$

$$\Rightarrow (-1)^1 \frac{d(f(s))}{ds} = L\{t F(t)\}$$

$\Rightarrow$ Theorem is true for $n=1$

Let us assume that thm is true for $n=k$.

$$\text{ie } (-1)^k \frac{d^k(f(s))}{ds^k} = L\{t^k F(t)\}$$

$$\Rightarrow (-1)^k \frac{d^k(f(s))}{ds^k} = \int_0^\infty e^{-st}\, t^k\, F(t)\,dt.$$

Diff. both sides w.r.to s.

$$\Rightarrow (-1)^k \frac{d^{k+1}(f(s))}{ds^{k+1}} = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) t^k\, F(t)\,dt$$

$$\Rightarrow (-1)^k \frac{d^{k+1}(f(s))}{ds^{k+1}} = \int_0^\infty e^{-st} (-t)\, t^k\, F(t)\,dt.$$

$$\Rightarrow (-1)^{k+1} \frac{d^{k+1}(f(s))}{ds^{k+1}} = \int_0^\infty e^{-st}\, t^{k+1} F(t)\,dt$$

$$\Rightarrow (-1)^{k+1} \frac{d^{k+1}(f(s))}{ds^{k+1}} = L\{t^{k+1} F(t)\}$$

$\Rightarrow$ It true for $n=k+1$. Hence by induction, the statement is true for all n.

Laplace transform of division by t

If $L\{F(t)\} = f(s)$ then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$.

Proof: Given $L\{F(t)\} = f(s)$

$$\Rightarrow f(s) = \int_0^\infty e^{-st} \cdot F(t)\,dt.$$

$$\Rightarrow \int_s^\infty f(s)\,ds = \int_s^\infty ds \int_0^\infty e^{-st} \cdot F(t)\,dt.$$

$$\Rightarrow \int_s^\infty f(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} \cdot F(t)\,ds$$

$$= \int_0^\infty F(t)\,dt \cdot \left[\frac{e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty -\frac{1}{t} F(t) \cdot \left\{0 - e^{-st}\right\} dt.$$

$$= \int_0^\infty e^{-st} \cdot \frac{F(t)}{t}\,dt$$

~~Ex Prove that:-~~

$$= L\left\{\frac{F(t)}{t}\right\}$$

Ex Prove that $F(t) = t^n$ is of exponential order as $t \to \infty$

Sol. We have $\lim_{x\to\infty} e^{-ax} f(x) = \lim_{x\to\infty} e^{-ax} \cdot x^n \quad a > 0$

$$= \lim_{x\to\infty} \frac{x^n}{e^{ax}}$$

$$= \lim_{x\to\infty} \frac{\lfloor n}{a^n e^{ax}} = \frac{\lfloor n}{\infty} = 0$$

$\Rightarrow t^n$ is function of exponential order as $x \to \infty$