

$$\therefore \lim_{T \rightarrow \infty} I = \frac{0+0}{1} = 0$$

Theorem state and prove complex inversion formula for Laplace Stieltjes integral Ex. 7.2

Statement: If $\alpha(t)$ is normalised function of b.v. in $(0, \infty)$, $\sigma > \sigma_c$ and the integral $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ has an abscissa of convergence σ_c then for $\sigma > 0$, $\sigma > \sigma_c$, an abscissa of convergence $\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{f(s)}{s} e^{st} ds = \begin{cases} \alpha(t), & t > 0 \\ \alpha(0+), & t = 0 \\ 0, & t < 0 \end{cases}$

Proof: To prove this theorem, we need to state the following Lemma

Lemma 1: If $\alpha(t)$ is of b.v. in $0 \leq t \leq S$, $S > 0$ then $\lim_{T \rightarrow \infty} \frac{1}{\pi} \int_0^S \alpha(t) \frac{\sin Tt}{t} dt = \frac{\alpha(0+)}{2}$

Lemma 2: If $\phi(u)$ belong to $L(-\infty, \infty)$ and is of b.v. in some two sided nbd of a point t , then $\lim_{T \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(u) \frac{\sin T(t-u)}{(t-u)} du = \frac{\phi(t-) + \phi(t+)}{2}$

Proof of Main theorem:

Let $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ has abscissa of convergence σ_c then for $\sigma > 0$, we have

$$f(s) = \int_0^\infty e^{-st} d\alpha(t) - \alpha(0)$$

$$= \int_0^\infty e^{-st} d\alpha(t)$$

$$\frac{f(s)}{s} = \int_0^\infty e^{-st} \alpha(t) dt$$

$\therefore \frac{f(s)}{s}$ is absolutely convergent for $c > 0, \sigma_c > 0$

Then, by complex inversion formulae for Laplace integral, we have

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{f(s)}{s} e^{st} ds = 0 \quad \text{if } t < 0$$

$$= \frac{\alpha(t+) + \alpha(t-)}{2}, \quad \text{if } t > 0$$

$$= \alpha(t) \quad \text{if } t > 0$$

$$= \frac{\alpha(0+) + \alpha(0-)}{2} \quad \text{if } t = 0$$

Q.10 Ans Q.1 what do you understand by an inversion formulae for Laplace transform

Laplace Transform can be classified into two class (1) Laplace integral (2) Laplace-Stieltjes integral

Complex inversion formulae for Laplace integral means the following

If the Laplace integral $f(s) = \int_0^{\infty} e^{-su} \phi(u) du$ is cgt. absolutely on the real line $\sigma = c$, then for $\phi(u) \in L(0, R)$, R being any +ve quantity

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) e^{st} ds = 0 \quad \text{when } t < 0$$

and for $\phi(u)$ being of b.v. in the nshd of $u = t > 0$

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) e^{st} ds = \frac{\phi(t-) + \phi(t+)}{2}, \quad t > 0$$

$$= \phi(0+) \quad , \quad t = 0$$

Complex inversion formulae for Laplace-Stieltjes integral as defined below.

If $L(t)$ is normalised function of b.v. in $(0, R)$, $R > 0$, and the integral $f(s) = \int_0^{\infty} e^{-st} dL(t)$ has an abscissa of cgt. σ_c then for $c > 0, c > \sigma_c$

$$11. f(s) = \frac{A}{s^\gamma} \rightarrow 0 \text{ as } s \rightarrow 0+$$

(61)

$$f(s) \sim \frac{A}{s^\gamma} \text{ as } s \rightarrow 0+$$

R.V.
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State and prove Abelian theorem / Abel's theorem

Statement: If $f(s) = \int_0^\infty e^{-st} d\alpha(t)$ is cgt for some $s > 0$, then $\lim_{s \rightarrow 0+} f(s) = \lim_{t \rightarrow \infty} \alpha(t)$, provided the

limit on the R.H.S exists and $\lim_{s \rightarrow \infty} f(s) = \alpha(0+)$

Proof:- To prove this theorem, we need to prove the following lemma.

Lemma: (1) State & prove lemma (2)

Proof of Main theorem

Let $\lim_{t \rightarrow \infty} \alpha(t)$, i.e. $\alpha(\infty)$ exists

Then by lemma (1)

$$\lim_{s \rightarrow 0+} |s^\gamma f(s) - A| \leq \lim_{t \rightarrow \infty} |\alpha(t) \Gamma(\gamma+1) t^{-\gamma} - A|$$

where $\gamma > 0$ & $A = \text{constant}$

Take $A = \alpha(\infty)$, $\gamma = 0$

$$\therefore \lim_{s \rightarrow 0+} |f(s) - \alpha(\infty)| \leq \lim_{t \rightarrow \infty} |\alpha(t) - \alpha(\infty)|$$

$$\therefore \lim_{s \rightarrow 0+} |f(s) - \alpha(\infty)| \leq 0 \quad \left\{ \because \lim_{t \rightarrow \infty} \alpha(t) \text{ exists} = \alpha(\infty) \right.$$

$$\Rightarrow \lim_{s \rightarrow 0+} |f(s) - \alpha(\infty)| = 0$$

$$\Rightarrow \lim_{s \rightarrow 0+} f(s) - \alpha(\infty) = 0 \text{ also}$$

$$\Rightarrow \lim_{s \rightarrow 0+} f(s) = \alpha(\infty) = \lim_{t \rightarrow \infty} \alpha(t)$$

Again by Lemma ②

$$\lim_{s \rightarrow \infty} |s^\gamma f(s) - A| \leq \lim_{t \rightarrow 0+} |\alpha(t) r(\gamma+1) t^\gamma - A|$$

Take $A = \alpha(0+)$, $\gamma = 0$

As before,

$$\lim_{s \rightarrow \infty} |f(s) - \alpha(0+)| \leq \lim_{t \rightarrow 0+} |\alpha(t) - \alpha(0+)|$$

$$\Rightarrow \lim_{s \rightarrow \infty} |f(s) - \alpha(0+)| \leq 0$$

$$\Rightarrow \lim_{s \rightarrow \infty} \{f(s) - \alpha(0+)\} = 0$$

$$\Rightarrow \lim_{s \rightarrow \infty} \{f(s) - \alpha(0+)\} = 0$$

$$\Rightarrow \lim_{s \rightarrow \infty} f(s) = \alpha(0+) = \lim_{t \rightarrow 0+} \alpha(t)$$

83 State and prove Tauberian Theorem for Laplace Integral
 Theorem: If $\int_0^\infty e^{-st} a(t) dt$ converges to sum $f(s)$ for $s > 0$
 i.e. if $f(s) = \int_0^\infty e^{-st} a(t) dt$ for $s > 0$, where $a(t) \in \mathcal{L}$
 $a(t) \in \mathcal{L}(0, \infty)$, f +ve R and $a(t) = o(\frac{1}{t})$ as $t \rightarrow \infty$
 then $f(0+) = \lim_{s \rightarrow 0+} f(s) = \int_0^\infty a(t) dt$

if \therefore we have $\int_0^\infty a(t) dt = \int_0^\infty a(u) du = \lim_{t \rightarrow \infty} \int_0^t a(u) du$

$$\therefore f(0+) = \lim_{s \rightarrow 0+} f(s) = \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} a(t) dt$$

$$= \lim_{s \rightarrow 0+} \int_0^\infty e^{-su} a(u) du$$

$$\text{Put } s = \frac{1}{t} \quad \therefore s \rightarrow 0+ \Rightarrow t = \infty$$

$$= \lim_{t \rightarrow \infty} \int_0^\infty e^{-\frac{u}{t}} a(u) du$$