

## FOURIER TRANSFORM (16)

Problem (11) find finite Fourier sine and cosine transform of  $f(x) = x^2$ ,  $0 < x < 4$ .

Answer (i)

$$F_s \{f(x)\} = \int_0^c f(x) \sin \frac{n\pi x}{c} dx$$
$$\Rightarrow F_s \{f(x)\} = \int_0^4 f(x) \sin \frac{n\pi x}{4} dx = \int_0^4 x^2 \sin \frac{n\pi x}{4} dx$$
$$\Rightarrow \bar{f}_s(n) = \left[ x^2 \frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_{x=0}^4 - \int_0^4 2x \cdot \frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} dx$$
$$= -\frac{4}{n\pi} \left[ -0 + 4^2 \cos n\pi \right] + \frac{8}{n\pi} \int_0^4 x \cos \frac{n\pi x}{4} dx$$
$$= -\frac{4^3 \cos n\pi}{n\pi} + \frac{8}{n\pi} \left[ x \cdot \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} - \int 1 \cdot \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_{x=0}^4$$
$$= -\frac{4^3 \cos n\pi}{n\pi} + \frac{8}{n\pi} \left[ \frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{4}{n\pi} \cdot \frac{\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_{x=0}^4$$
$$= -\frac{4^3 \cos n\pi}{n\pi} + \frac{8}{n\pi} \left[ \frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{16}{n^2 \pi^2} \cos \frac{n\pi x}{4} \right]_{x=0}^4$$
$$= -\frac{64}{n\pi} \cos n\pi + \frac{8}{n\pi} \left[ \frac{16}{n\pi} \sin n\pi + \frac{16}{n^2 \pi^2} \cos n\pi - \left( 0 + \frac{16}{n^2 \pi^2} \cdot 1 \right) \right]$$
$$= -\frac{64}{n\pi} \cos n\pi + \frac{8}{n\pi} \left[ 0 + \frac{16}{n^2 \pi^2} (\cos n\pi - 1) \right]$$
$$\bar{f}_s(n) = -\frac{64}{n\pi} \cos n\pi + \frac{128}{n^3 \pi^3} (\cos n\pi - 1) \quad \underline{\text{Ans.}}$$



$$(ii) F_c \{f(x)\} = \bar{f}_c(n) = \int_0^4 f(x) \cos \frac{n\pi x}{4} dx$$

$$= \int_0^4 x^2 \cos \frac{n\pi x}{4} dx$$

$$= \left[ x^2 \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_{x=0}^4 - \int_0^4 2x \cdot \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} dx$$

$$= \left[ \frac{4x^2}{n\pi} \sin \frac{n\pi x}{4} \right]_{x=0}^4 - \int_0^4 \frac{8x}{n\pi} \sin \frac{n\pi x}{4} dx$$

$$= \left( \frac{64}{n\pi} \sin n\pi - 0 \right) - \frac{8}{n\pi} \left[ x \cdot \frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} - \int 1 \cdot \frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}} dx \right]_{x=0}^4$$

$$= \frac{64}{n\pi} \cdot 0 + \frac{8}{n\pi} \cdot \frac{4}{n\pi} \left[ x \cos \frac{n\pi x}{4} \right]_0^4 - \frac{8}{n\pi} \cdot \frac{4}{n\pi} \left[ \frac{\sin \frac{n\pi x}{4}}{\frac{n\pi}{4}} \right]_0^4$$

$$= \frac{32}{n^2 \pi^2} [4 \cos n\pi - 0 \cos 0] - \frac{128}{n^3 \pi^3} [\sin n\pi - \sin 0]$$

$$= \frac{128}{n^4 \pi^2} \cos n\pi \quad \underline{\underline{\text{Ans.}}}$$

Problem (12) find the finite sine transform of  
(i)  $e^{ax}$  (ii)  $\sin ax$  (iii)  $\cos ax$  (iv)  $x^3$

Answer  $\bar{f}_s(s) = \int_0^\pi f(x) \sin\left(\frac{s\pi x}{\pi}\right) dx = \int_0^\pi f(x) \sin(sx) dx$

$$(i) F_s \{e^{ax}\} = \int_0^\pi e^{ax} \sin sx dx$$

$$= \left[ \frac{e^{ax} (a \sin sx - s \cos sx)}{a^2 + s^2} \right]_{x=0}^\pi$$

Ans. H.C.C.



$$\begin{aligned}
 F_s \{e^{ax}\} &= \frac{1}{a^2 + s^2} \left[ e^{ax} (a \sin sx - s \cos sx) \right]_{x=0}^{\pi} \\
 &= \frac{1}{a^2 + s^2} \left[ e^{a\pi} (a \sin s\pi - s \cos s\pi) - e^0 (a \sin 0 - s \cos 0) \right] \\
 &= \frac{1}{a^2 + s^2} \left[ e^{a\pi} (0 - s \cos s\pi) + s \cdot 1 \right] \\
 &= \frac{1}{a^2 + s^2} \left[ s - s \cdot e^{a\pi} \cos s\pi \right] \\
 &= \frac{s}{a^2 + s^2} \left[ 1 - e^{a\pi} \cdot (-1)^s \right] \quad \underline{\text{Ans.}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } F_s \{ \sin ax \} &= \int_0^{\pi} \sin ax \cdot \sin sx \, dx \\
 &= \frac{1}{2} \int_0^{\pi} 2 \sin ax \sin sx \, dx \\
 &= \frac{1}{2} \int_0^{\pi} \{ \cos(a-s)x - \cos(a+s)x \} \, dx \\
 &= \frac{1}{2} \left[ \frac{\sin(a-s)x}{a-s} - \frac{\sin(a+s)x}{a+s} \right]_{x=0}^{\pi} \\
 &= \frac{1}{2} (0 - 0) = 0, \text{ if } a \neq s \text{ and } a, s \text{ are integers.}
 \end{aligned}$$

Also if  $s = a$ , then

$$\begin{aligned}
 F_s \{ \sin ax \} &= \frac{1}{2} \int_0^{\pi} 2 \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi} (1 - \cos 2sx) \, dx \\
 &= \frac{1}{2} \left[ x - \frac{\sin 2sx}{2s} \right]_0^{\pi} = \frac{\pi}{2}
 \end{aligned}$$



$$\therefore F_s \{ \sin an \} = \begin{cases} 0, & \text{if } a \neq s \text{ and } a, s \text{ are integers} \\ \frac{\pi}{2}, & \text{if } a = s. \end{cases}$$

Ans

$$\begin{aligned} \text{(iii)} \quad F_s \{ \cos an \} &= \int_0^\pi \cos an \sin sn \, dn \\ &= \frac{1}{2} \int_0^\pi 2 \sin sn \cos an \, dn \\ &= \frac{1}{2} \int_0^\pi [\sin(s+a)n + \sin(s-a)n] \, dn \\ &= \frac{1}{2} \left[ \frac{-\cos(s+a)n}{s+a} - \frac{\cos(s-a)n}{s-a} \right]_0^\pi \\ &= -\frac{1}{2(s+a)} \left[ \cos(s+a)\pi + \cos(s-a)\pi \right]_{n=0}^\pi \\ &= -\frac{1}{2(s+a)} \left[ \cos(s+a)\pi + \cos(s-a)\pi - (\cos 0 + \cos 0) \right] \\ &= -\frac{1}{2(s+a)} \left[ \cos(s\pi + a\pi) + \cos(s\pi - a\pi) - 2 \right] \\ &= -\frac{1}{2(s+a)} \left[ 2 \cos s\pi \cos a\pi - 2 \right] \\ &= -\frac{1}{s+a} \left[ (-1)^s \cos a\pi - 1 \right] \\ &= \frac{1}{s+a} \left[ 1 - (-1)^s \cos a\pi \right], \text{ if } s \text{ is +ve.} \end{aligned}$$

As



$$\begin{aligned}
\text{(iv)} \quad F_3(x^3) &= \int_0^{\pi} x^3 \sin xs \, dx \\
&= \left( -x^3 \frac{\cos xs}{s} \right)_0^{\pi} - \int_0^{\pi} 3x^2 \cdot \frac{-\cos xs}{s} \, dx \\
&= \left( -\frac{\pi^3}{s} \cos \pi s - 0 \right) + \frac{3}{s} \int_0^{\pi} x^2 \cos xs \, dx \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{3}{s} \left[ x^2 \frac{\sin xs}{s} - \int 2x \cdot \frac{\sin xs}{s} \, dx \right]_0^{\pi} \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{3}{s^2} \left[ x^2 \sin xs \right]_0^{\pi} - \frac{6}{s^2} \int_0^{\pi} x \sin xs \, dx \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{3}{s^2} (0-0) - \frac{6}{s^2} \left[ x \left( -\frac{\cos xs}{s} \right) - \int 1 \cdot \frac{-\cos xs}{s} \, dx \right]_0^{\pi} \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{6}{s^3} \left[ x \cos xs \right]_0^{\pi} - \frac{6}{s^3} \left[ \frac{\sin xs}{s} \right]_{x=0}^{\pi} \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{6}{s^3} \left[ \pi \cos \pi s - 0 \right] - \frac{6}{s^4} [0-0] \\
&= -\frac{\pi^3}{s} \cos \pi s + \frac{6\pi}{s^3} \cos \pi s \\
&= \frac{\pi}{s} \cos \pi s \left( \frac{6}{s^2} - \pi^2 \right) \\
&= \frac{\pi}{s} (-1)^s \left\{ \frac{6}{s^2} - \pi^2 \right\}.
\end{aligned}$$

Ans

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