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Banach Space

PG sem III
Paper CC-308

Theorem: Open mapping Theorem

Statement: Let B and B' be Banach spaces. If T is a continuous linear transformation of B onto B', then T is an open mapping.

Proof: It is given that linear transformation

$$T : B \to B'$$

is continuous and onto and we want to show that T is an open mapping, i.e $T[G]$ is an open set in B' for every open set G in B.

Let $y \in T(G)$ is an arbitrary. Then $y = T(x)$ for some $x \in G$. Since G is an open set in B, there exists an open sphere $S(x,r)$ in B centered at x such that $S(x,r) \subset G$. But we can write $S(x,r) = x + S_r$, where S_r is an open sphere in B centred at origin.

$$\text{Thus } x + S_r \subset U \quad \text{——— (1)}$$

But by lemma we have i.e let B and B' be Banach spaces and T a continuous linear transformation of B onto B'. Then the image of each open sphere centred on the origin in B contains an open sphere centered on the origin in B'.

Thus there exists an open sphere S_ϵ in B'

centered at origin s.t. $S'_\epsilon \subset T[S_r]$

$\therefore \; y + S'_\epsilon \subset y + T[S_r] = T(x) + T[S_r]$

$= T[x + S_r]$

$\Rightarrow S'(y, \epsilon) \subset T[x + S_r] \quad [\because y + S'_\epsilon = S'(y,\epsilon)]$

$\subset T(G)$ by (1)

Thus we have shown that to each $y \in T[G]$, there exists an open sphere in B' centered at y and contained in $T[G]$ and consequently $T[G]$ is an open set, this completing the proof of theorem.

Theorem: The closed graph Theorem.

Statement: Let B and B' be Banach spaces and let T be a linear transformation of B into B'. Then T is a continuous mapping if and only if its graph is closed.

Proof: The 'only if' part

Let T be continuous and let T_G be the graph of T. Now we shall show that $\overline{T} = T_G$ and this will prove that T_G is closed. Since $T_G \subset \overline{T_G}$ always, so we need only prove $\overline{T_G} = T_G$. Let $(x, y) \in \overline{T_G}$. Then (x, y) is an adherent point of T_G. Hence there exists a sequence $\langle x_n, T(x_n) \rangle$

in T_G s.t. $(x_n, T(x_n)) \to (x, y)$
which implies that $x_n \to x$ and $T(x_n) \to y$.
But, T is continuous, $x_n \to x \Rightarrow T(x_n) \to T(x)$
and so $y = T(x)$. This shows that
$(x, y) = (x, T(x)) \in T_G$ so $\overline{T_G} \subset T_G$.

<u>The 'if' Part</u> let T_G be closed. We denote by B_1 the linear space B renormed by
$$\|x\|_1 = \|x\| + \|T(x)\|.$$
Now $\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_1$.
which shows that T is bounded and consequently continuous. It therefor sufficient to show that B and B_1 have the same topology. That is they are homomorphic.
let us consider the identity map
$$I : B_1 \to B : I(x) = x \ \forall \ x \in B_1.$$
Clearly I is One-one onto.
Further $\|I(x)\| = \|x\| \leq \|x\| + \|T(x)\| = \|x\|_1$
which shows that I is bounded and hence continuous. If we can show that B_1 is complete, then by theorem "If B and B' be Banach spaces and let T be a One-one

continuous linear transformation of B onto B'. Then T is a homomorphism". That means T is a homomorphism and this will complete the proof.

So let $\langle x_n \rangle$ be an Cauchy sequence in B_1 so that

$$\|x_n - x_m\|_I \to 0 \text{ as } m, n \to \infty.$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \to 0, \ m, n \to \infty$$

$$\Rightarrow \|x_n - x_m\| \to 0 \text{ and } \|T(x_n) - T(x_m)\| \to 0 \text{ as } m, n \to \infty$$

$\Rightarrow \langle x_n \rangle$ is a cauchy sequence in B and $\langle T(x_n) \rangle$ is a Cauchy sequence in B'.

Since B and B' are complete, we have, $x_n \to x \in B$ and $T(x_n) \to y \in B'$. (I)

Since the graph of T_G is given to be closed. (I) shows that

$(x, y) \in T_G$ so that $y = T(x)$.

$$\begin{aligned} \text{Now } \|x_n - x\|_I &= \|x_n - x\| + \|T(x_n - x)\| \\ &= \|x_n - x\| + \|T(x_n) - T(x)\| \\ &= \|x_n - x\| + \|T(x_n) - y\| \\ &\to 0 \quad (\because x_n \to x, \ T(x_n) \to y) \end{aligned}$$

It follows that the sequence $\langle x_n \rangle$ in B_1 converges to $x \in B_1$ and hence B_1 is complete as required.

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Contd.

I am also sending you the Proof of Hahn-Banach Theorem.

Also Generalized Hahn-Banach Theorem

Note: you prepare Hahn-Banach Theorem. it is generally asked in university Question

In the Proof of Hahn-Banach Theorem it is totally based on Proof of a Lemma. After then the main Theorem is proved

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Theorem 1. (Hahn-Banach Theorem). *Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. Then f can be extended to a functional F defined on the whole space N such that*

$$\|F\|=\|f\|.$$

[M.U. 1973, 72 (Statement) 76]

Proof. We first prove the following lemma which constitutes the most difficult part of the theorem.

Lemma. *Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If $x_0 \notin M$ and if*

$$M_0=(M\cup\{x_0\})=\{x+\alpha x_0 : x \in M, \alpha \text{ real}\}$$

is the linear subspace spanned by M and x_0, then f can be extended to a functional f_0 defined on M_0 such that

$$\|f_0\|=\|f\|.$$

Proof of the lemma. We prove the lemma for real and complex scalars separately.

Case I. Let N be a real normed space. [M. U. 1977]

Since x_0 is not in M, each vector w in M_0 is uniquely expressible in the form $w=x+\alpha x_0$ with $x \in M$. We define our f_0 by setting $f_0(w)=f_0(x+\alpha x_0)=f(x)+\alpha r_0$ ...(1)

where r_0 is any real number. It is easy to see that for every choice of the real number r_0, f_0 is linear on M_0 such that

$$f_0(x)=f(x) \text{ for all } x \in M.$$

[For if $\beta, \gamma \in \mathbf{R}$ and $x, y \in M$, then $f_0(\beta(x+\alpha x_0)+\gamma(y+\alpha x_0))$

$$=f_0(\beta x+\gamma y+(\beta+\gamma)\alpha x_0)$$
$$=f(\beta x+\gamma y)+(\beta+\gamma)\alpha r_0$$
$$=\beta f(x)+\gamma f(y)+\beta\alpha r_0+\gamma\alpha r_0$$
$$=\beta(f(x)+\alpha r_0)+\gamma(f(y)+\alpha r_0)$$
$$=\beta f_0[(x+\alpha x_0)+\gamma f_0(y+\alpha x_0)]$$

Thus f_0 extends f linearly to M_0. We now prove that

$$\| f_0 \| = \| f \| .$$

We have

$$\| f_0 \| = \sup \{ | f_0(x) | : x \in M_0, \| x = \leqslant 1 \}$$
$$\geqslant \sup \{ | f_0(x) | : x \in M, \| x \| \leqslant 1 \} \quad [\because \; M_0 \supset M]$$
$$= \sup \{ | f(x) | : x \in M, \| x = \leqslant 1 \} \quad [\because \; f_0 = f \text{ on } M]$$
$$= \| f \| .$$

Thus $\quad \| f_0 \| \geqslant \| f \| \qquad \ldots$(A)

So our problem now is to choose r_0 such that $\| f_0 \| \leqslant \| f \|$.
For this purpose, we first observe that if x_1, x_2 are any two vectors in M, then

$$f(x_2) - f(x_1) = f(x_2 - x_1) \leqslant | f(x_2 - x_1) |$$
$$\leqslant \| f \| \; \| x_2 - x_1 \|$$
$$= \| f \| \; \| (x_2 + x_0) - (x_1 + x_0) \|$$
$$\leqslant \| f \| (\| x_2 + x_0 \| + \| -(x_1 + x_0) \|)$$
$$= \| f \| \; \| x_2 + x_0 \| + \| f \| \; \| x_1 + x_0 \|$$

Thus $\quad -f(x_1) - \| f \| \; \| x_1 + x_0 \| \leqslant -f(x_2) + \| f \| \; \| x_2 + x_0 \|$

Since this inequality holds for arbitrary $x_1, x_2 \in M$, we see that

$$\sup_{y \in M} \{ -f(y) - \| f \| \; \| y + x_0 | \} \leqslant \inf_{y \in M} \{ -f(y) + \| f \| \; \| y + x_0 \| \}$$

Choose r_0 to be any real number such that

$$\sup_{y \in M} \{ -f(y) - \| f \| \; \| y + x_0 \| \} \leqslant r_0 \leqslant \inf_{y \in M} \{ -f(y) + \| f \| \; \| y + x_0 \| \}$$

It follows that

$$-f(y) - \| f \| \; \| y + x_0 \| \leqslant r_0 \leqslant -f(y) + \| f \| \; \| y + x_0 \| \; \forall \; y \in M \qquad \ldots(2)$$

With this choice of r_0, we shall prove that $\| f_0 \| \leqslant \| f \|$.
Let $w = x + \alpha x_0$ be an arbitrary vector in M_0.

Putting $y = \dfrac{x}{\alpha}$ in (2), we get

$$-f\left(\frac{x}{\alpha}\right) - \| f \| \left| \frac{x}{\alpha} + x_0 \right| \leqslant r_0 \leqslant -f\left(\frac{x}{\alpha}\right) + \| f \| \left\| \frac{x}{\alpha} + x_0 \right\| \qquad \ldots(3)$$

If $\alpha > 0$, then right hand inequality in (3) gives,

$$r_0 \leqslant -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} \| f \| \; \| x + \alpha x_0 \|$$
$$\Rightarrow f(x) + \alpha r_0 \leqslant \| f \| \; \| x + \alpha x_0 \|$$
$$\Rightarrow f_0(x + \alpha x_0) \leqslant \| f \| \; \| x + \alpha x_0 \|$$

$$\Rightarrow f_0(w) \|f\| \; \|w\|.$$

If $\alpha < 0$, we use left hand inequality in (3) to obtain

$$r_0 \geqslant -f\left(\frac{x}{\alpha}\right) - \|f\| \left\|\frac{x}{\alpha}+x_0\right\|$$

$$= -\frac{1}{\alpha} f(x) - \|f\| \left|\frac{1}{\alpha}\right| \|x+x_0\alpha\|$$

$$= -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} \|f\| \; \|x+x_0\alpha\| \qquad \left[\because \; \alpha<0 \Rightarrow \left|\frac{1}{\alpha}\right| = -\frac{1}{\alpha}\right]$$

We now multiply both sides of the above inequality by α. Since $\alpha < 0$, the inequality will be reversed. Hence we obtain

$$\alpha r_0 \leqslant -f(x) + \|f\| \; \|x+x_0\alpha\|$$

or $$f(x)+\alpha r_0 \leqslant \|f\| \; \|x+x_0\alpha\|$$

or $$f_0(w) \leqslant \|f\| \; \|w\| \quad \text{by (1)}$$

Thus we have shown that when $\alpha \neq 0$, then

$$f_0(w) \leqslant \|f\| \; \|w\| \; \forall \; w \in M_0 \qquad \ldots(3)$$

[Of course when $\alpha=0$, $\|f_0\| = \|f\|$]

Replacing w by $-w$, we get

$$f_0(-w) \leqslant \|f\| \; \|-w\| \; \text{ i.e. } -f_0(w) \leqslant \|f\| \; \|w\|. \qquad \ldots(4)$$

From (3) and (4), we conclude that

$$|f_0(w)| \leqslant \|f\| \; \|w\| \qquad \ldots(5)$$

[Since f is bounded (being a linear functional), (5) shows that f_0 is bounded and so f_0 is a linear functional on M_0].

Since $\|f_0\| = \sup\{|f_0(w)| : w \in M_0, \; \|w\| \leqslant 1\}$,
it follows from (5) that $\|f_0\| \leqslant \|f\|$. .. (B)

From (A) and (B), we finally obtain $\|f_0\| = \|f\|$.
This proves the lemma for real scalars.//

Case II. Let N be a complex normed linear space.

Let N be a normed linear space over **C** and let f be a complex-valued linear functional defined on a subspace M of N.

Let $g=\text{Re} f$ and $h=\text{Im} f$ so that $f(x)=g(x)+ih(x)$ for every $x \in M$. Then an easy computation shows that g and h are real-valued functionals on the real space M.

$[f(x+y)]=f(x)+f(y)$

$$\Rightarrow g(x+y)+ih(x+y)=g(x)+ih(x)+g(y)+ih(y)$$
$$\Rightarrow g(x+y)=g(x)+g(y) \text{ and } h(x+y)=h(x)+h(y).$$

and if $\alpha \in R$, then

$$f(\alpha x)=\alpha f(x) \Rightarrow g(\alpha x)+ih(\alpha x)=\alpha\,[g(x)+ih(x)]$$

$$\Rightarrow g(\alpha x)=\alpha g(x) \text{ and } h(\alpha x)=\alpha h(x).$$

Thus g and h are linear on M. Further

$$|g(x)| \leqslant |f(x)| \leqslant \|f\| \; \|x\|$$

and so boundedness of f implies boundedness of g and similarly of h. Hence g, h are real linear functionals on the real space M]

Also for all $x \in M$, we have

$$g(ix)+ih(ix)=f(ix = if(x)=-h(x)+ig(x)$$

whence equating real and imaginary parts,

$$g(ix)=-h(x) \text{ and } h(ix)=g(x).$$

Consequently, $f(x)=g(x)-ig(ix)=h(ix)+ih(x)$.

Let $f(x)=g(x)-ig(ix)$.

Since g is a real-valued functional on the real space M, by Case I, g can be extended to a real-valued functional g_0 on the real space M_0 in such a way that $\|g_0\|=\|g\|$.

We now define f_0 for $x \in M_0$ by $f_0(x)=g_0(x)-ig_0(ix)$. It is easy to see that f_0 is linear on the complex space M_0 such that

$$f_0=f \text{ on } M.$$

$$\begin{aligned}[f_0(x+y) &= g_0(x+y)-ig_0(ix+iy)=g_0(x)+g_0(y) \\ &\qquad\qquad -ig_0(ix)-ig_0(iy) \\ &= g_0(x)-ig_0(ix)+g_0(y)-ig_0(iy) \\ &= f_0(x)+f_0(y).\end{aligned}$$

And if $a, b \in \mathbf{R}$, then

$$\begin{aligned} f_0((a+ib)x) &= g_0(ax+ibx)-ig_0(-bx+iax), \text{ by def. of } f_0 \\ &= ag_0(x)+bg_0(ix)-i(-b)g_0(x)-iag_0(ix) \\ &= (a+ib)[g_0(x)-ig_0(ix)] \\ &= (a+ib)f_0(x).\end{aligned}$$

Thus f_0 is linear on M_0. Also $g_0=g$ on M implies $f_0=f$ on M].

What remains to prove is that $\|f_0\|=\|f\|$.

Let $x \in M_0$ be arbitrary and write $f_0(x)=re^{i\theta}$ where $r \geqslant 0$ and θ real. Then

$$\begin{aligned} |f_0(x)| &= r = e^{-i\theta} \cdot re^{i\theta} = e^{-i\theta} f_0(x) = f_0(e^{-i\theta} x) \\ &= g_0(e^{-i\theta} x) \qquad [\because \; r \text{ is real}] \\ &\leqslant |g_0(e^{-i\theta} x)| \leqslant \|g_0\| \; \|e^{-i\theta} x\| \\ &= \|g_0\| \; |e^{-i\theta}| \; \|x\| = \|g_0\| \; \|x\| \quad [\because \; |e^{-i\theta}|=1] \\ &= \|g\| \; \|x\| \leqslant \|f\| \; \|x\|.\end{aligned}$$

This shows that f_0 is bounded (hence a functional on M_0) and that $\|f_0\| \leqslant \|f\|$. Also as in Case I, it is obvious that $\|f\| \leqslant \|f_0\|$.

Therefore $\|f_0\|=\|f\|$.

This completes the proof of the lemma.

Proof of the main theorem. If $M_0=N$, then we finish ; if not, we may repeat the process of extension, but what guarantee is there that we shall ever extend to the whole space N ? It is here that we need Zorn's lemma which states :

'Every non-empty partially ordered set in which each chain has an upper bound has a maximal element'.

Let P denote the set of all ordered pairs (f_λ, M_λ) where f_λ is an extension of f to the subspace $M_\lambda \supset M$ and $\|f_\lambda\|=\|f\|$. Partially order P by setting $(f_\lambda, M_\lambda) \leqslant (f_\mu, M_\mu)$ iff $M_\lambda \subset M_\mu$ and $f_\lambda=f_\mu$ on M_λ [The reader can easily verify that $\leqslant$ is actually a partial ordering on P]. P is evidently non-empty, for certainly $(f, M) \in P$, and further, by virtue of the lemma, it is seen that there are less trivial members of P. Let $Q=\{(f_i, M_i)\}$ be a chain (*i.e.* a totally ordered set) in P. Then it is easy to see that Q has an upper bound $(\varphi, \cup M_i)$ where $\varphi(x)=f_i(x)$ for all $x \in M_i$. The point to be noted here is that $\cup M_i$ is a subspace of N and that φ is well defined because of total ordering on Q

[Let $x, y \in \cup M_i$ and α, β any scalars. Then for some i, j, we have $x \in M_i$ and $y \in M_j$. Since Q is totally ordered, either $M_i \subset M_j$ or $M_j \subset M_i$. Without loss of generality, we may assume $M_i \subset M_j$. Then $x, y \in M_i$. Since M_i is a subspace of N, we have $\alpha x+\beta y \in M_i \subset \cup M_i$ showing that $\cup M_i$ is a subspace of N. To show that φ is well-defined, suppose an element x in $\cup M_i$ is such that $x \in M_i$ and $x \in M_j$. Then by the definition of φ, we have $\varphi(x)=f_i(x)$ and $\varphi(x)=f_j(x)$. By total ordering of Q. either f_i extends f_j or vice versa. In either case, $f_i(x)=f_j(x)$. Thus φ is well-defined].

Now all the conditions of Zorn's lemma are satisfied. Hence there exists a maximal element (F, H) in P. To complete the proof, we must show that $H=N$. Suppose, if possible, N contains H properly. Then there exists $x_0 \in N-H$ and so by our lemma, F can be extended to a functional F_0 on

$$H_0=(H \cup \{x_0\})$$

which contains H properly. But this contradicts the maximality of (F, H). Consequently, we must have $H=N$ and the proof is complete.

Note 1. Explanations in square brackets [ ] may be omitted by the students in the examination.

Sublinear functionals and the generalized Hahn-Banach Theorem.

Definition. *Let L be a linear space. A mapping*

$$p : L \to \mathbf{R}$$

is called a **sublinear functional** *on L if it satisfies the following two properties*

(*i*) $p(x+y) \leqslant p(x)+p(y) \ \forall \ x, y \in L$,

(*ii*) $p(\alpha x)=\alpha p(x)$ *provided* $\alpha \geqslant 0$.

The property (i) is called **subadditivity** and (ii) the **positive-homogeneity**.

Illustration. Let $L=\mathbf{R}^n$. If $x=(\alpha_1,..., \alpha_n) \in \mathbf{R}^n$, let us define $p(x)=|\alpha_1|+...+|\alpha_n|$. Then p is a sublinear functional on L.

Convex Functional.

If in addition, p satisfies the condition

(iii) $p(x) \geqslant 0 \ \forall \ x \in N$,

then p is called a **convex functional.**

A convex functional p is said to be **symmetric** if we have

(iv) $p(\alpha x)=|\alpha| p(x)$ for all scalars α

Note 2. As in remark 1 of § 1, the condition (iii) can be deduced from the conditions (i) and (iv). Thus p will be a symmetric convex functional if it satisfies (i) and (iv)

Theorem 2. **(Generalized Hahn-Banach Theorem).** *Let L be a real linear space, not necessarily normed and let p be a sublinear functional on L, i.e, p is a map from L into* R *satisfying*

$$p(x+y) \leqslant p(x)+p(y) \text{ for all } x \in L \text{ and all } \alpha \geqslant 0 \quad ...(1)$$

$$p(\alpha x)=\alpha p(x) \text{ for all } x \in L \text{ and all } \alpha > 0 \quad ...(2)$$

If f is a real linear functional defined on a linear subspce M such that $f(x) \leqslant p(x)$ *for all x in M, then there exists a real linear function F defined on the whole space L such that* $f=F$ *on M and* $F(x) \leqslant p(x)$ *for all* $x \in N$.

If L is complex linear space, then condition (1) *is the same but* (2) *is modified to* $p(\alpha x)=|\alpha| p(x)$ *for all* $x \in L$ *and scalars* α ...(2)'

And f is a complex linear functional on M such that $|f(x)| \leqslant p(x)$ *for all* $x \in M$.

The conclusion in this case is the same except that we have

$$|F(x)| \leqslant p(x) \ \forall \ x \in L.$$

Proof. Case I. First let L be a real linear space.

If $x_n \notin M$, consider the subspace

$$M_0=(M \cup \{x_0\})=\{x+\alpha x_0 : x \in M, \alpha \text{ real}\}$$

spanned by M and x_0. Define f_0 on M_0 by $f_0(x+\alpha x)=f(x)+\alpha r_0$ where r_0 is any real number so that f_0 is real value. It is easy to see that f_0 is linear on M_0 and $f_0 = f$ on M. If x_1, x_2 are any vectors in M, then

$$f(x_2)-f(x_1)=f(x_2-x_1) \leqslant p(x_2-x_1) \text{ by hypothesis}$$
$$=p((x_2+x_0)-(x_1+x_0)) \leqslant p(x_2+x_0)$$
$$+p(-x_1-x_0) \text{ by (1)}$$

so that $\qquad -f(x_1)-p(-x-x_0) \leqslant -f(x_2)+p(x_2+x_0)$

Since this inequality holds for arbitrary $x_1, x_2 \in M$, we conclude that

$$\sup_{y\in M}\{-f(y)-p(-y-x_0)\} \leqslant r_0 \leqslant \inf_{y\in M}\{-f(y)+p(y+x_0)\}$$

Choose r_0 to be any real number such that

$$\sup_{y\in M}\{-f(y)-p(-y-x_0)\} \leqslant r_0 \leqslant \inf_{y\in M}\{-f(y)+p(y+x_0)\}.$$

It follows that

$$-f(y)-p(-y-x_0) \leqslant r_0 \leqslant \{-f(y)+p(y+x_0)\} \qquad \ldots(3)$$

for all $y \in M$. With this choice of r_0, we shall show that

$$f_0(x) \leqslant p(x) \text{ for all } x \in M_0.$$

Let $w = x+\alpha x_0$ be an arbitrary element in M_0. If $\alpha = 0$, then

$$f_0(w) = f(x) \leqslant p(x).$$

So let $\alpha \neq 0$ and put $y = \frac{x}{\alpha}$ in (3) to obtain

$$-f\left(\frac{x}{\alpha}\right)-p\left(-\frac{x}{\alpha}-x_0\right) \leqslant r_0 \leqslant -f\left(\frac{x}{\alpha}\right)+p\left(\frac{x}{\alpha}+x_0\right) \qquad \ldots(4)$$

for all $x \in M$. If $\alpha > 0$, then the right hand inequality in (4) gives

$$r_0 \leqslant -\frac{1}{\alpha} f(x_0)+\frac{1}{\alpha} p(x+\alpha x_0)$$

$$\Rightarrow f(x)+\alpha r_0 \leqslant p(x+\alpha x_0) \Rightarrow f_0(x+\alpha x_0) \leqslant p(x+\alpha x_0).$$

And if $\alpha < 0$, then the left hand inequality in (4) gives,

$$r_0 \geqslant -f\left(\frac{x}{\alpha}\right)-p\left(-\frac{x}{\alpha}-x_0\right) = -f\left(\frac{x}{\alpha}\right)-p\left(-\frac{1}{\alpha}\left(x+\alpha x_0\right)\right)$$

$$= -\frac{1}{\alpha} f(x)-\left(-\frac{1}{\alpha}\right) p(x+\alpha x_0) \text{ by (2) since } -\frac{1}{\alpha} > 0$$

$$= -\frac{1}{\alpha} f(x)+\frac{1}{\alpha} p(x+\alpha x_0)$$

We now multiply both sides of this inequality by α. Since $\alpha < 0$, the inequality will be reversed.

$$\therefore \quad \alpha r_0+f(x) \leqslant p(x+\alpha x_0) \Rightarrow f_0(x+\alpha x_0) \leqslant p(x+\alpha x).$$

Thus when $\alpha \neq 0$, we obtain

$$f_0(x+\alpha x_0) \leqslant p(x+\alpha x_0) \text{ for all } x \in M$$

i.e. $f_0(w) \leqslant p(w)$ for all $w \in M_0$. Thus f_0 is a real linear functional on M_0 such that $f_0(x) = f(x)$ for all $x \in M$ and $f_0(w) \leqslant p(w)$ for all $w \in M_0$.

If $M_0 = L$, then we finish : if not we may repeat the process of extension ; but what guarantee is there that we shall ever extend to the whole space L. It is at this point that Zorn's lemma is needed. Let P denote the set of all ordered pairs (f_λ, M_λ) where f_λ is an extension of f to the subspace $M_\lambda \supset M$ and

$$f_\lambda(x) \leqslant p(x) \text{ for all } x \in M_\lambda.$$

Partially order P by setting $(f_\lambda, M_\lambda) \leqslant (f_\mu, M_\mu)$ iff $M_\lambda \subset M_\mu$ and $f_\lambda = f_\mu$ on M_λ. P is evidently non-empty. Let $Q = \{f_\iota, M_\iota)\}$ be a chain (*i.e* a totally ordered set) in P. Then it is easy to see that Q has an upper bound.

$$(\varphi, \cup M_\iota) \text{ where } \varphi(x) = f_\iota(x) \text{ for all } x \in M_\iota.$$

The point to be noted is that $\cup M_\iota$ is a subspace of N and that φ is well-defined because of total ordering on φ

[For proof, see the previous theorem]

Hence by Zorn's lemma, P contains a maximal element (F, H). To complete the proof, we must show that $H = N$ Suppose, if possible, N contains H properly. Then there exists $x_0 \in N - H$ and by first part of the theorem, F can be extended to a functional F_0 on $H_0 = (H \cup \{x_0\})$ which contains H properly. But this contradicts the maximality of (F, H). Consequently, we must have $H = N$ and the proof is complete.

Case II. Now let L be a complex linear space.

Here f is a complex linear functional on M such that

$$|f(x)| \leqslant p(x) \text{ for all } x \in M.$$

Let $f_1 = \text{Re} f$, then $f_1(x) \leqslant |f(x)| \leqslant p$ and so by case II, f_1 can be extended to a linear map F_1 of L into $\mathbf{R}$ such that $F_1 = f_1$ on M and $F_1(x) \leqslant p(x)$ for all $x \in L$. Define F by

$$F(x) = F_1(x) - i F_1(ix), x \in L.$$

Then it is easy to see that F is a linear functional on L such that $F = f$ on M. What remains to prove is that $|F(x)| \leqslant p(x)$ for all $x \in L$. Let $x \in L$ be arbitrary and write $F(x) = re^{i\theta}$ where $r > 0$ and θ is real. Then

$$\begin{aligned}
|F(x)| &= r = e^{-i\theta}.re^{i\theta} = e^{-i\theta} F(x) = F(e^{-i\theta} x) \\
&= F_1(e^{-i\theta} x) \qquad [\because r \text{ is real}] \\
&\leqslant p(e^{-i\theta} x) \text{ since } F_1(x) \leqslant p(x) \ \forall \ x \in L \\
&= |e^{-i\theta}| \, p(x) \text{ by } (2)' \\
&= p(x)
\end{aligned}$$