Q: Show that $L\{(1+te^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}.$

Sol^n We know that $L\{t^n\} = \frac{\lfloor n}{s^{n+1}}$ if n is +ve integer

Also $L\{t^n . e^{at}\} = \frac{\lfloor n}{(s-a)^{n+1}}$ (Using shifting th^m).

$$L\{(1+te^{-t})^3\} = L\{1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\}$$

$$= L\{1\} + 3L\{t.e^{-t}\} + 3L\{t^2e^{-2t}\} + L\{t^3e^{-3t}\}$$

$$= \frac{1}{s} + 3.\frac{\lfloor 1}{(s+1)^2} + 3\frac{\lfloor 2}{(s+2)^3} + \frac{\lfloor 3}{(s+3)^4}$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Ans

Q: Find $L\{\sin\sqrt{t}\}$.

Sol^n We know that $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \cdots$

$$\Rightarrow \sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \cdots$$

$$\Rightarrow L\{\sin\sqrt{t}\} = L\{t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \cdots\}$$

$$= L\{t^{1/2}\} - \frac{1}{\lfloor 3} \cdot L\{t^{3/2}\} + \frac{1}{\lfloor 5} L\{t^{5/2}\} - \cdots$$

$$= \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{1}{\lfloor 3}\frac{\Gamma\frac{5}{2}}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\Gamma\frac{7}{2}}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\cdot\sqrt{\pi}}{s^{3/2}} - \frac{1}{\lfloor 3}\cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \cdots$$

$$\left[\because \Gamma(n+1) = n\Gamma n \ \& \ \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{s^{3/2}}\cdot\left[\frac{1}{2} - \frac{1}{\lfloor 3}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{\lfloor 5}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}}\left[1 - \frac{1}{2^2 s} + \frac{1}{(2^2 s)^2 \lfloor 2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{2^2 s}} \quad \underline{\underline{\text{Ans}}}$$

Q: Find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Let $F(t) = \sin\sqrt{t}$

$\Rightarrow F(0) = 0$

$f(s) = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{4s}}$

We know that

$$L\{F'(t)\} = s f(s) - F(0)$$

$$\Rightarrow L\left\{\frac{d}{dt}(\sin\sqrt{t})\right\} = s\cdot\frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot e^{-\frac{1}{4s}} - 0$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-\frac{1}{4s}}.$$

$$\Rightarrow \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-\frac{1}{4s}}$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$ Ans

Ex: Show that
(i) $L\{\sinh at \cdot \cos at\} = \dfrac{a(p^2-2a^2)}{p^4+4a^4}$
(ii) $L\{\sinh at \cdot \sin at\} = \dfrac{2a^2p}{p^4+4a^4}$.

Sol^n We know that

$$L\{\sinh at\} = \frac{a}{p^2-a^2}$$

$$\Rightarrow L\{e^{iat} \cdot \sinh at\} = \frac{a}{(p-ia)^2-a^2} \quad \text{(Shifting theorem)}$$

$$= \frac{a}{(p^2-2a^2)-2pia}$$

$$= \frac{a\{(p^2-2a^2)+2ipa\}}{(p^2-2a^2)^2+(2pa)^2} \quad \text{(Rationalise)}$$

$$= \frac{a(p^2-2a^2)+2ia^2p}{p^4+4a^4}$$

$$= \frac{a(p^2-2a^2)}{p^4+4a^4} + i\,\frac{2a^2p}{p^4+4a^4} \quad \text{---(1)}$$

Also $L\{e^{iat} \cdot \sinh at\} = L\{(\cos at + i\sin at)\sinh at\}$
$= L\{\cos at \cdot \sinh at\} + i\, L\{\sin at \cdot \sinh at\}$ ---(2)

Equating real parts of ① & ②

$$L\{\sinh at.\cos at\} = \frac{a(p^2-2a^2)}{p^4+4a^4}.$$

& equating imaginary parts of ① & ②

$$L\{\sinh at \sin at\} = \frac{2a^2p}{p^4+4a^4}.$$

Ans

Q: Find $L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\}$

Soln We know that if $L\{F(t)\} = f(s)$

then $L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s} f(s)$ ——— ①

Now $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L(\sin t)\,ds$

$$= \int_s^\infty \frac{1}{1+s^2}\,ds \quad \left(\because L\{\sin at\} = \frac{a}{a^2+s^2}\right)$$

$$= \left(\tan^{-1} s\right)_s^\infty$$

$$= \pi/2 - \tan^{-1} s$$

$$= \cot^{-1} s.$$

$$\Rightarrow L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\} = \frac{1}{s}\cot^{-1} s \quad (\text{Using prop. (1)})$$

Q: Find $L\{t \cdot J_0(at)\}$

Sol^n

$$L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2\cdot 2} + \frac{t^4}{2\cdot 4\cdot 2\cdot 4} - \cdots\right\}$$

$$= \left\{\frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{4!}{2^2\cdot 4^2\cdot s^5} - \cdots\right\}$$

$$= \frac{1}{s}\left\{1 - \left(\frac{1}{2}\right)\frac{1}{s^2} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}\left(\frac{1}{s^2}\right)^2 - \cdots\right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2}$$

$$= \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{-1/2}$$

$$= \frac{1}{s}\cdot\frac{s}{(s^2+1)^{1/2}}$$

$$= \frac{1}{\sqrt{s^2+1}}$$

$$L\{J_0(at)\} = \frac{1}{a}\cdot\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{1}{\sqrt{s^2+a^2}} \qquad \left(\because L\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)\right)$$

$$\Rightarrow L\{t\, J_0(at)\} = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)$$

$$= -\frac{-1}{\cancel{2}(s^2+a^2)^{3/2}}\cdot \cancel{2}s = \frac{s}{(s^2+a^2)^{3/2}}$$

Ans