Ex. The error function is defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\,dx$$

Prove that $L\{\operatorname{erf}(\sqrt{t})\} = \dfrac{1}{s\sqrt{s+1}}$

Proof

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - x^2 + \frac{(x^2)^2}{\underline{|2}} - \frac{(x^2)^3}{\underline{|3}} + \cdots\right] dx$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\,\underline{|2}} - \frac{t^{7/2}}{7\cdot\underline{|3}} + \cdots\right]$$

$$L\{\operatorname{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}} L\left\{t^{1/2} - \frac{t^{3/2}}{3\,\underline{|1}} + \frac{t^{5/2}}{5\,\underline{|2}} - \frac{t^{7/2}}{7\,\underline{|3}} + \cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{s^{5/2}\cdot 3\,\underline{|1}} + \frac{\Gamma(7/2)}{s^{7/2}\,5\,\underline{|2}} - \frac{\Gamma(9/2)}{s^{9/2}\,7\,\underline{|3}} + \cdots\right]$$

$$\left[\because L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}\cdot 3\,\underline{|1}} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}\,5\,\underline{|2}} - \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{9/2}\,7\,\underline{|3}} + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{s^3} + \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{s^3} + \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s^{3/2}} \frac{s^{1/2}}{(s+1)^{1/2}} = \frac{1}{s\sqrt{s+1}} \quad \text{Ans}$$

EX: Find $L\{F(t-a)\}$ where $F(t-a)$ is Heaviside's unit step function

Sol^n We know that

$$F(t-a) = \begin{cases} 1 & \text{if } t>a \\ 0 & \text{if } t<a. \end{cases}$$

Now $$L\{F(t-a)\} = \int_0^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^a e^{-st} F(t-a)\,dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt.$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= -\frac{1}{s}\left(0 - e^{-sa}\right)$$

$$= \frac{e^{-sa}}{s} \quad \text{Ans}$$

INVERSE LAPLACE TRANSFORM

If the laplace transform of a function $F(t)$ is $f(s)$, then inverse Laplace transform of $f(s)$ is $F(t)$.

If $L\{F(t)\} = f(s)$ then $L^{-1}\{f(s)\} = F(t)$

or.

If $L\{F(t)\} = f(p)$ then $L^{-1}\{f(p)\} = F(t)$

Null function

Let $N(t)$ be a function defined as

$$\int_0^{t_0} N(t)\,dt = 0 \quad \forall\ t_0 > 0$$

then $N(t)$ is called null function.

Ex Prove that Laplace transform of Null function is zero.

Sol^n

$$L\{N(t)\} = \int_0^\infty e^{-st} \cdot N(t)\, dt.$$

~~We have~~ $\cancel{\int e^{-st} N(t)\, dt = e^{-st} \cdot \underbrace{\int N(t)\, dt}_{0} \mp \int \left\{ -se^{-st} \cdot \underbrace{\int N(t)\, dt}_{0} \right\}}$

$$= 0.$$

Uniqueness of Inverse Laplace transform

If $L\{F(t)\} = f(s)$ ——— (1)

then $L\{F(t) + N(t)\} = f(s) + 0 = f(s)$ ——— (2)

So we can $L^{-1}\{f(s)\} = F(t)$ (From (1))

& $L^{-1}\{f(s)\} = F(t) + N(t)$

$\Rightarrow$ Inverse Laplace transform of a functions is not unique

Lerch's theorem: Let $L\{F(t)\} = f(s)$. Let $F(t)$ be piecewise continuous in every interval $0 \leq t \leq a$ and is of exponential order for $t > a$, then the inverse Laplace transform of $f(s)$ is unique

Standard.

$L^{-1}\left\{\frac{1}{s}\right\} = 1$ $\quad \boxed{\because L\{1\} = \frac{1}{s}}$

$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\lfloor n}$ $\quad \forall\, n = 0, 1, 2, \ldots$ $\quad \boxed{\because L\{t^n\} = \frac{\lfloor n}{s^{n+1}}}$

$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at.$

$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

$L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$