Sohagm
K.C.C

Theorem (6) Define covariant derivative of covariant tensor of second order and show that it is a covariant tensor of rank three.

Proof: Let A_{ij} be a second rank covariant tensor, then by tensor law of transformation

$$A'_{iJ} = A_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J}$$

Differentiating w.r.t. x'^k, we have

$$\frac{\partial A'_{iJ}}{\partial x'^K} = \frac{\partial A_{ab}}{\partial x^c} \frac{\partial x^c}{\partial x'^K} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} + A_{ab} \frac{\partial^2 x^a}{\partial x'^i \partial x'^k} \frac{\partial x^b}{\partial x'^J} + A_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial^2 x^b}{\partial x'^J \partial x'^k} \quad \text{--- (1)}$$

Now $$A_{ab} \frac{\partial^2 x^a}{\partial x'^i \partial x'^k} \frac{\partial x^b}{\partial x'^J} = A_{pb} \frac{\partial^2 x^p}{\partial x'^i \partial x'^k} \frac{\partial x^b}{\partial x'^J}$$

$$= A_{pb} \left[\Gamma'^{r}_{ik} \frac{\partial x^p}{\partial x'^r} - \Gamma^p_{ac} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^c}{\partial x'^k} \right] \frac{\partial x^b}{\partial x'^J}$$

$$\left[\because \Gamma'^{p}_{iJ} \frac{\partial x^c}{\partial x'^p} = \Gamma^c_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} + \frac{\partial^2 x^c}{\partial x'^i \partial x'^J} \right]$$

$$\therefore A_{ab} \frac{\partial^2 x^a}{\partial x'^i \partial x'^k} \frac{\partial x^b}{\partial x'^J} = A_{pb} \frac{\partial x^p}{\partial x'^r} \frac{\partial x^b}{\partial x'^J} \Gamma'^{r}_{ik} - A_{pb} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} \frac{\partial x^c}{\partial x'^k} \Gamma^p_{ab}$$

$$= A'_{rJ} \Gamma'^{r}_{ik} - A_{pb} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} \frac{\partial x^c}{\partial x'^k} \Gamma^p_{ab} \quad \text{--- (2)}$$

and $$A_{ab} \frac{\partial x^a}{\partial x'^i} \frac{\partial^2 x^b}{\partial x'^J \partial x'^k} = A_{ap} \frac{\partial x^a}{\partial x'^i} \left[\Gamma'^{r}_{Jk} \frac{\partial x^p}{\partial x'^r} - \Gamma^p_{bc} \frac{\partial x^b}{\partial x'^J} \frac{\partial x^c}{\partial x'^k} \right]$$

$$= A_{ap} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^p}{\partial x'^r} \Gamma'^{r}_{JK} - A_{ap} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} \frac{\partial x^c}{\partial x'^k} \Gamma^p_{bc}$$

$$= A'_{ir} \Gamma'^{r}_{JK} - A_{ap} \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^J} \frac{\partial x^c}{\partial x'^k} \Gamma^p_{bc} \quad \text{--- (3)}$$

Now writing (1) with the help of (2) and (3), we have

$$\frac{\partial A'_{ij}}{\partial x'^k} = \frac{\partial A_{ab}}{\partial x^c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k} + A'_{rj}\Gamma'^r_{ik}$$
$$- A_{pb}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Gamma^p_{ab} + A'_{ir}\Gamma'^r_{jk}$$
$$- A_{ap}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Gamma^p_{bc}$$

$$\Rightarrow \frac{\partial A'_{ij}}{\partial x'^k} - A'_{rj}\Gamma'^r_{ik} - A'_{ir}\Gamma'^r_{jk}$$
$$= \left[\frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ab} - A_{ap}\Gamma^p_{bc}\right]\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$

If we write $A_{ab,c} = \frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ac} - A_{ap}\Gamma^p_{bc}$

Then the last becomes

$$A'_{ij,k} = A_{ab,c}\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}$$

It proves that it is a third rank covariant tensor and this tensor is defined as covariant of A_{ab} w. r. t. x^c.

Definition: Covariant differentiation of a tensor:

The covariant differentiation of a tensor is denoted by a subscript preceded by a comma or semicolon.

i.e. $A_{a;b} = A_{a,b} = \frac{\partial A_a}{\partial x^b} - A_c\Gamma^a_{bc}$

$$A_{ab,c} = \frac{\partial A_{ab}}{\partial x^c} - A_{pb}\Gamma^p_{ac} - A_{ap}\Gamma^p_{bc}.$$

K.C.C.

Theorem (7) Show that covariant derivatives of the fundamental tensor and Kronecker delta vanish. Hence define covariant constants.

Proof: We have to prove that

$$g_{ij,k} = 0 \quad \text{--- (1)}$$

$$g^{ij}{}_{,k} = 0 \quad \text{--- (2)}$$

$$\delta^{i}_{j,k} = 0 \quad \text{--- (3)}$$, Here δ^i_j is Kronecker delta.

$$\because \quad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{aj}\Gamma^a_{ik} - g_{ia}\Gamma^a_{jk}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik,j} - \Gamma_{jk,i}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \left[\Gamma_{ik,j} + \Gamma_{jk,i}\right]$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} \quad \text{(By Property of christoffel)}$$

$= 0$, Hence the equation (1).

Now $g_{ij}g^{jk} = \delta^k_i = 1 \text{ or } 0$

Differentiang it w.r.t. x^m,

$$g_{ij}\frac{\partial g^{jk}}{\partial x^m} + \frac{\partial g_{ij}}{\partial x^m}g^{jk} = 0$$

Multiplying both sides by g^{li} and noting that $g_{ij}g^{li} = \delta^l_j$, $\delta^l_j \frac{\partial g^{jk}}{\partial x^m} = \frac{\partial g^{lk}}{\partial x^m}$

Now we obtain, $g^{li}g^{jk}\frac{\partial g_{ij}}{\partial x^m} + g^{li}g_{ij}\frac{\partial g^{jk}}{\partial x^m} = 0$

Sattagm
K.C.C.

$$\Rightarrow g^{li} g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \delta^l_j \frac{\partial g^{jk}}{\partial x^m} = 0$$

$$\Rightarrow g^{li} g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{lk}}{\partial x^m} = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} g^{jk} \left[\Gamma_{im,j} + \Gamma_{jm,i} \right] = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} \Gamma^k_{im} + g^{jk} \Gamma^l_{jm} = 0$$

$$\Rightarrow \frac{\partial g^{lk}}{\partial x^m} + g^{li} \Gamma^k_{im} + g^{ik} \Gamma^l_{im} = 0$$

$$\Rightarrow g^{lk}_{,m} = 0 \quad \text{or} \quad g^{ij}_{,k} = 0$$

Hence proof of equation 2.

Now $$\delta^i_{j,k} = \frac{\partial \delta^i_j}{\partial x^k} + \delta^m_j \Gamma^i_{mk} - \delta^i_a \Gamma^a_{jk}$$

$$= \frac{\partial \delta^i_j}{\partial x^k} + \Gamma^i_{jk} - \Gamma^i_{jk}$$

$$= \frac{\partial \delta^i_j}{\partial x^k} = 0 \quad \text{as } \delta^i_j = 1 \text{ or } 0.$$

Second part we have seen that Covariant derivatives of all fundamental tensors vanish identically. Hence they act as constants relative to covariant differentiation. Hence the tensors g_{ij}, g^{ij}, g^i_j are defined as covariant constants.

R, State and Prove Ricci's theorem