(Change of scale property)

Theorem: If $f(s) = L\{F(t)\}$ then $L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$, $a > 0$

Proof

$$f(s) = \int_0^\infty e^{-st} \cdot F(t)\, dt.$$

$$\Rightarrow f(as) = \int_0^\infty e^{-ast} F(t)\, dt.$$

Let $at = u$

$\Rightarrow a\,dt = du$

& $dt = \frac{du}{a}$

When $t = 0$ $u = 0$ & when $t = \infty$ $u = \infty$

$$= \int_0^\infty e^{-su} \cdot F\left(\frac{u}{a}\right) \cdot \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-su} \cdot F\left(\frac{u}{a}\right) du$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{a} F\left(\frac{t}{a}\right) dt$$

$$= L\left\{\frac{1}{a} F\left(\frac{t}{a}\right)\right\}$$

ie $f(as) = L\left\{\frac{1}{a} F\left(\frac{t}{a}\right)\right\}$

$\Rightarrow L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$ Ans

Q: If $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, then find $L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\}$

Soln Since $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$

$\Rightarrow L^{-1}\left[\frac{(3s)^2-1}{\{(3s)^2+1\}^2}\right] = \frac{1}{3} \frac{t}{3} \cos\frac{t}{3} = \frac{t}{9} \cos\frac{t}{3}$ Ans

(Using change of scale property)

Inverse Laplace transform of derivatives:

Th^m If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{d^n f(s)}{ds^n}\right\} = (-1)^n t^n F(t)$.

Proof We know that

If $L\{F(t)\} = f(s)$

then $L\{t^n F(t)\} = (-1)^n f^{(n)}(s)$ where $f^{(n)}(s) = \frac{d^n f(s)}{ds^n}$

$$\Rightarrow L^{-1}\{(-1)^n f^{(n)}(s)\} = t^n F(t)$$

$$\Rightarrow (-1)^n L^{-1}\{f^{(n)}(s)\} = t^n F(t)$$

$$\Rightarrow L^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t) = (-1)^n t^n L^{-1}\{f(s)\}$$

proved.

Ex Find $L^{-1}\left\{\log\left(\frac{p+3}{p+2}\right)\right\}$

Sol^n Let $f(p) = \log\left(\frac{p+3}{p+2}\right)$

$$\Rightarrow f(p) = \log(p+3) - \log(p+2)$$

$$\Rightarrow f'(p) = \frac{1}{p+3} - \frac{1}{p+2}$$

$$\Rightarrow L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+3}\right\} - L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$= e^{-3t} L^{-1}\left\{\frac{1}{p}\right\} - e^{-2t} L^{-1}\left\{\frac{1}{p}\right\}$$

$$= e^{-3t} \cdot 1 - e^{-2t} \cdot 1$$

$$= e^{-3t} - e^{-2t} \quad \text{——— (1)}$$

But $L^{-1}\{f'(p)\} = (-1)^1 t^1 L^{-1}\{f(p)\} = -t L^{-1}\{f(p)\}$

Putting in (1)

$$-t L^{-1}\{f(p)\} = e^{-3t} - e^{-2t}$$

$$\Rightarrow L^{-1}\{f(p)\} = -\frac{1}{t}\left(e^{-3t} - e^{-2t}\right)$$

$$= \frac{1}{t}\left(e^{-2t} - e^{-3t}\right)$$

Ans

Th^m (Division by s): If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$

Sol^n We know that

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s} \quad \text{when } f(s) = L\{F(t)\}$$

$$\Rightarrow L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du.$$

Ex Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol^n We know that

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at.$$

$$\Rightarrow L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\} = (-1)^1 \cdot t^1 \cdot \frac{1}{a}\sin at.$$

$$\Rightarrow L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = -\frac{t}{a}\sin at$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at \quad \text{Ans}$$

Ex Find $L^{-1}\left\{\frac{1}{s^2 \cdot (s+1)^2}\right\}$

Sol^n We have

$$L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \cdot L^{-1}\left\{\frac{1}{s}\right\} = e^{-t} \cdot 1 = e^{-t}.$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\Rightarrow L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+1}\right)\right\} = (-1)^{1} \cdot t^{1} \cdot e^{-t}$$

$$\Rightarrow L^{-1}\left\{\frac{-1}{(s+1)^2}\right\} = -t e^{-t}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t \cdot e^{-t}.$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t u e^{-u}\, du. \quad \text{(Division by s)}$$

$$= \left[-u e^{-u} - e^{-u}\right]_0^t$$

$$= -(t e^{-t} - 0) - (e^{-t} - 1)$$

$$= -t e^{-t} - e^{-t} + 1$$

Again dividing by s

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t \left(-u e^{-u} - e^{-u} + 1\right) du$$

$$= \left[u e^{-u} + e^{-u} + e^{-u} + u\right]_0^t$$

$$= (t \cdot e^{-t} - 0) + (e^{-t} - 1) + (e^{-t} - 1) + (t - 0)$$

$$= t e^{-t} + 2 e^{-t} + t - 2 \quad \text{Ans}$$