

1<sup>st</sup> Convolution Theorem: If  $L^{-1}\{f(s)\} = F(t)$  and  $L^{-1}\{g(s)\} = G(t)$  (1)  
 then  $L^{-1}\{f(s) \cdot g(s)\} = \int_0^t F(u) G(t-u) du = \int_0^t F(t-u) \cdot G(u) du = F * G$ .

Sol<sup>n</sup> We have to prove that

$$L\left\{\int_0^t F(u) \cdot G(t-u) du\right\} = f(s) \cdot g(s).$$

Now let  $\int_0^t F(u) G(t-u) du = H(t)$ .

So we have to show that

$$L\{H(t)\} = f(s) \cdot g(s).$$

Now  $L\{H(t)\} = \int_0^\infty e^{-st} H(t) dt$ .

$$= \int_0^\infty e^{-st} \int_0^t F(u) G(t-u) du dt$$

$$= \int_0^\infty \int_0^t e^{-st} F(u) G(t-u) du dt$$

We now change the order of integration.

We have limits  $t=0, t=\infty$  &  $u=0, u=t$ .

Which is given by following diagram (1)

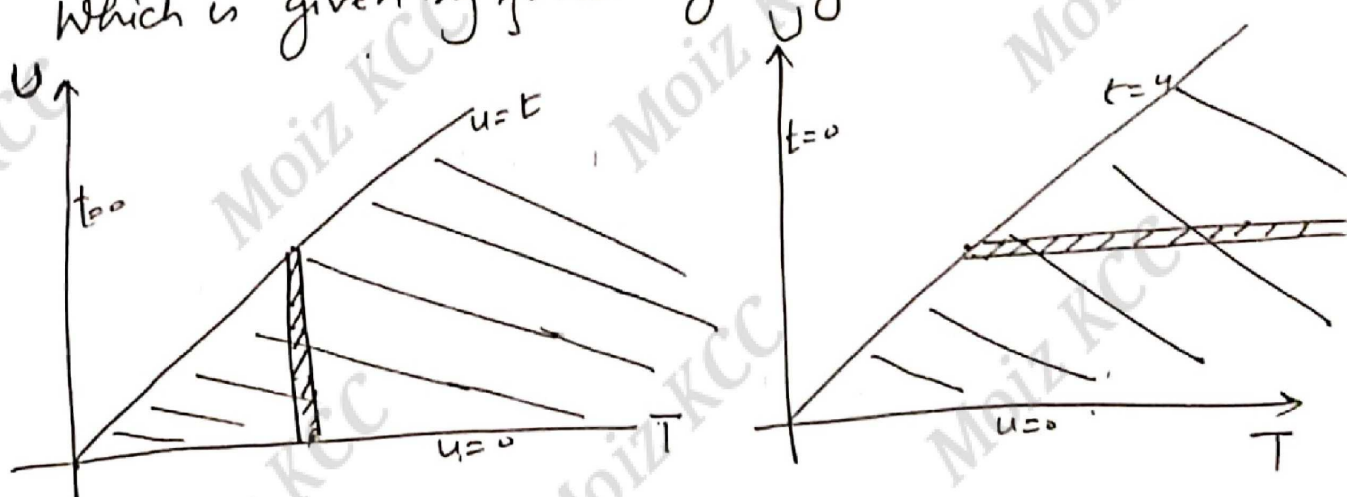


Fig (1)

Now we take horizontal ( $\parallel$  to  $T$ -axis) strip. New limit will be  $t=u, t=\infty$  &  $u=0, u=\infty$

Now the integral becomes

$$= \int_0^{\infty} \int_u^{\infty} e^{-st} F(u) G(t-u) dt du$$

$$= \int_0^{\infty} F(u) du \int_u^{\infty} e^{-st} G(t-u) dt$$

$$\left. \begin{aligned} \text{Let } t-u &= v \\ \Rightarrow dt &= dv \end{aligned} \right\} \begin{aligned} \text{When } t=u & \quad v=0 \\ \text{When } t=\infty & \quad v=\infty \end{aligned}$$

$$= \int_0^{\infty} F(u) du \int_0^{\infty} e^{-s(u+v)} G(v) dv$$

$$= \int_0^{\infty} e^{-su} F(u) du \cdot \int_0^{\infty} e^{-sv} G(v) dv$$

$$= f(s) \cdot g(s)$$

Ex Find  $L^{-1} \left\{ \frac{1}{(s-1)\sqrt{s}} \right\}$  Proof

We have  $L^{-1} \left\{ \frac{1}{s-1} \right\} = e^t$       $L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} = \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$

$$L^{-1} \left\{ \frac{1}{s-1} \cdot \frac{1}{\sqrt{s}} \right\} = \int_0^t \frac{1}{\sqrt{\pi u}} e^{t-u} du$$

$$= \frac{e^t}{\sqrt{\pi}} \int_0^t \frac{e^{-u}}{\sqrt{u}} du$$

$$= \frac{e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{e^{-v^2}}{v} \cdot 2v dv$$

$$= \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv = e^t \cdot \text{erf}(\sqrt{t})$$

$$\left\{ \begin{aligned} &\text{substitute} \\ &\text{Let } u = v^2 \\ &du = 2v dv \\ &\sqrt{u} = v \\ &\text{When } t=0 \quad v=0 \\ &\quad \quad \quad t=t \quad v=\sqrt{t} \end{aligned} \right.$$

Ex Find  $L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\}$ .

(3)

Sol Let  $f(s) = \frac{1}{s+a}$  &  $g(s) = \frac{1}{s+b}$

$$\Rightarrow F(t) = L^{-1} \{ f(s) \} = L^{-1} \left\{ \frac{1}{s+a} \right\} = e^{-at}$$

$$G(t) = L^{-1} \{ g(s) \} = L^{-1} \left\{ \frac{1}{s+b} \right\} = e^{-bt}$$

$$L^{-1} \{ f(s) \cdot g(s) \} = \int_0^t F(u) \cdot G(t-u) du$$

$$= \int_0^t e^{-au} \cdot e^{-b(t-u)} du$$

$$= e^{-bt} \int_0^t e^{-au} \cdot e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du$$

$$= e^{-bt} \left[ \frac{e^{(b-a)u}}{b-a} \right]_{u=0}^{u=t}$$

$$= \frac{e^{-bt}}{b-a} \left[ e^{(b-a)t} - e^{(b-a) \cdot 0} \right]$$

$$= \frac{e^{-bt}}{b-a} \left[ e^{(b-a)t} - 1 \right]$$

$$= \frac{1}{b-a} \left[ e^{-at} - e^{-bt} \right]$$

Ans