Tensor Calculus 18

Theorem (8) Show that the covariant differentiation for products, sums, differences obeys the same rule as in the case of ordinary differentiation:

Proof: (i) Let A_{ij} and B_{ij} be covariant tensors of rank two.

Let $C_{ij} = A_{ij} + B_{ij}$ ——— (1)

Since sum of two tensors is a tensor of the same rank and similar character. Hence C_{ij} is a second rank covariant tensor.

$$C_{ij,k} = \frac{\partial C_{ij}}{\partial x^k} - C_{\alpha j}\Gamma^{\alpha}_{ik} - C_{i\alpha}\Gamma^{\alpha}_{jk}$$

$$\Rightarrow C_{ij,k} = \frac{\partial}{\partial x^k}(A_{ij} + B_{ij}) - (A_{\alpha j} + B_{\alpha j})\Gamma^{\alpha}_{ik} - (A_{i\alpha} + B_{i\alpha})\Gamma^{\alpha}_{jk} \quad \text{by (1)}$$

$$= \frac{\partial A_{ij}}{\partial x^k} + \frac{\partial B_{ij}}{\partial x^k} - A_{\alpha j}\Gamma^{\alpha}_{ik} - B_{\alpha j}\Gamma^{\alpha}_{ik} - A_{i\alpha}\Gamma^{\alpha}_{jk} - B_{i\alpha}\Gamma^{\alpha}_{jk}$$

$$= \left(\frac{\partial A_{ij}}{\partial x^k} - A_{\alpha j}\Gamma^{\alpha}_{ik} - A_{i\alpha}\Gamma^{\alpha}_{jk}\right) + \left(\frac{\partial B_{ij}}{\partial x^k} - B_{\alpha j}\Gamma^{\alpha}_{ik} - B_{i\alpha}\Gamma^{\alpha}_{jk}\right)$$

i.e. $(A_{ij} + B_{ij})_{,k} = A_{ij,k} + B_{ij,k}$ ——— (2)

Similarly we can show that

(ii) $(A_{ij} - B_{ij})_{,k} = A_{ij,k} - B_{ij,k}$ ——— (3)

(iii) Let A^i_j be a mixed tensor and B_k is a covariant vector. Let $(P^i_{jk}) = A^i_j B_k$ ——— (4)

Since open product of two tensors is a tensor and hence P^i_{JK} is a tensor.

$$(A^i_J B_K)_{,l} = P^i_{JK,l}$$

$$= \frac{\partial P^i_{JK}}{\partial x^l} + P^\alpha_{JK} \Gamma^i_{\alpha l} - P^i_{\alpha k} \Gamma^\alpha_{Jl} - P^i_{J\alpha} \Gamma^\alpha_{kl}.$$

$$(A^i_J B_K)_{,l} = \frac{\partial (A^i_J B_K)}{\partial x^l} + (A^\alpha_J B_K) \Gamma^i_{\alpha l} - (A^i_\alpha B_K) \Gamma^\alpha_{Jl}$$

$$- (A^i_J B_\alpha) \Gamma^\alpha_{kl} \quad \text{by } (4)$$

$$= \left(\frac{\partial A^i_J}{\partial x^l} + A^\alpha_J \Gamma^i_{\alpha l} - A^i_\alpha \Gamma^\alpha_{Jl} \right) B_K + A^i_J \left(\frac{\partial B_K}{\partial x^l} - B_\alpha \Gamma^\alpha_{Kl} \right)$$

$$= (A^i_{J,l}) B_K + A^i_J B_{K,l}$$

ie $$(A^i_J B_K)_{,l} = A^i_{J,l} B_K + A^i_J B_{K,l} \quad \text{——} (5)$$

Hence (2), (3) and (5) are required results.

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