Theorem (9) Let ϕ and ψ be scaler functions of Coordinates x^i. Let A be an arbitrary vector. Then

(i) $div(\phi A) = \phi \, div A + A \cdot \nabla \phi$

(ii) $\nabla(\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$

(iii) $\nabla^2 \phi\psi = \phi \nabla^2 \psi + \psi \nabla^2 \phi + \nabla\phi \cdot \nabla\psi$

(iv) $div(\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla\phi \cdot \nabla\psi$.

Proof (i) We know that

$$div B^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} B^i) \quad \text{--- (1)}$$

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Proof (i) We know that

$$\operatorname{div} B^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, B^i) \quad \text{--- (1)}$$

Where B^i being an arbitrary contravariant vector.

Taking $B^i = \phi A^i$,

$$\operatorname{div}(\phi A^i) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\phi A^i \sqrt{g})$$

$$= \frac{1}{\sqrt{g}} \frac{\partial \phi}{\partial x^i} . A^i \sqrt{g} + \frac{1}{\sqrt{g}} \frac{\partial (A^i \sqrt{g})}{\partial x^i} \phi$$

$$= A^i \frac{\partial \phi}{\partial x^i} + \frac{\phi}{\sqrt{g}} \frac{\partial (A^i \sqrt{g})}{\partial x^i}$$

$$= A . \nabla \phi + \phi \operatorname{div} A^i \quad \text{by (1)}$$

$$\therefore \operatorname{div}(\phi A) = \phi \operatorname{div} A + A . \nabla \phi \quad \text{--- (2)}$$

(ii) $$\nabla(\phi \psi) = \frac{\partial}{\partial x^i}(\phi \psi) = \frac{\partial \phi}{\partial x^i} \psi + \phi \frac{\partial \psi}{\partial x^i}$$

$$= \psi \nabla \phi + \phi \nabla \psi \quad \text{--- (3)}$$

$$\therefore \nabla(\phi \psi) = \psi \nabla \phi + \phi \nabla \psi. \text{ Proved}$$

(iii) First prove the results (i) and (ii)

Taking divergence of both sides of (3)

$$\nabla . \nabla(\phi \psi) = \nabla . (\psi \nabla \phi) + \nabla . (\phi \nabla \psi)$$

$$\nabla^2(\phi \psi) = \operatorname{div}(\psi \nabla \phi) + \operatorname{div}(\phi \nabla \psi)$$

$$= [\psi \operatorname{div} \nabla \phi + \nabla \phi . \nabla \psi] +$$

$$[\phi \operatorname{div} \nabla \psi + \nabla \psi . \nabla \phi]$$

$= \psi \nabla \cdot \nabla \phi + \nabla \phi \cdot \nabla \psi + \phi \nabla \cdot \nabla \psi + \nabla \phi \cdot \nabla \psi$

$= \psi \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi$

Finally, $\nabla^2 \phi \psi = \phi \nabla^2 \psi + \psi \nabla^2 \phi + 2 \nabla \phi \cdot \nabla \psi$ Proved.

(iv) First prove the result (i)

Replacing A by $\nabla \psi$ in (2)

$\text{div}(\phi \nabla \psi) = \phi \,\text{div} \nabla \psi + \nabla \psi \cdot \nabla \phi$

$= \phi \nabla \cdot \nabla \psi + \nabla \psi \cdot \nabla \phi$

$= \phi \nabla^2 \psi + \nabla \psi \cdot \nabla \phi$

Interchanging ϕ and ψ

$\text{div}(\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi$ Proved.

Theorem (10) Laplacian operator ∇^2. If ϕ is a scalar function of co-ordinates x^i, then

$$\text{div grad}\, \phi = \nabla^2 \phi = g^{ij}\left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} - \frac{\partial \phi}{\partial x^l}\left\{ {l \atop ij} \right\}\right)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \phi_{,j}\right) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j \phi\right).$$

Proof: $\because \nabla \phi = \dfrac{\partial \phi}{\partial x^i} = \phi_{,i}$ ——— (1)

writing $\phi_{,i} = \phi_i$ ——— (2)

$\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{div} \nabla \phi = \text{div} \phi_i$

$= g^{ij} \phi_{i,j}$ (by definition of divergence

$\therefore\ \nabla^2\phi = g^{ij}\phi_{i,j} = g^{ij}(\phi_{,i})_{,j} = g^{ij}\phi_{,ij}$ Sattam K.C.L

finally, we have $\nabla^2\phi = g^{ij}\phi_{i,j} = g^{ij}\phi_{,ij}$ — (3)

$$\phi_{i,j} = \frac{\partial\phi_i}{\partial x^j} - \phi_a\begin{Bmatrix} a \\ ij \end{Bmatrix} = \frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^i}\right) - \frac{\partial\phi}{\partial x^a}\begin{Bmatrix} a \\ ij \end{Bmatrix}$$

$$\phi_{,ij} = \frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{\partial\phi}{\partial x^a}\begin{Bmatrix} a \\ ij \end{Bmatrix} \quad \text{— (4)}$$

using (3), we get

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{\partial\phi}{\partial x^a}\begin{Bmatrix} a \\ ij \end{Bmatrix}\right) \quad \text{— (5)}$$

Also we know that

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} A_j\right) = \operatorname{div} A_i = \operatorname{div} A^i$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, A^i\right)$$

$\therefore\ \nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\cdot\phi_i$ by (1) and (2)

$$= \operatorname{div}\phi_i = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\phi_j\right)$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\phi_{,j}\right)$$

$$\therefore\ \nabla^2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\phi_{,j}\right) \quad \text{— (6)}$$

Combing (5) and (6), we have

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i\partial x^j} - \frac{\partial\phi}{\partial x^l}\begin{Bmatrix} l \\ ij \end{Bmatrix}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\phi_{,j}\right)$$

Proved