Ex Solve using Laplace transform.

$$y''(t) + y(t) = t. \quad \text{Given } y'(0) = 1 \quad y(\pi) = 0$$

Soln Given differential eqn is

$$y'' + y = t.$$

$$\Rightarrow L\{y''\} + L\{y\} = L\{t\}$$

$$\Rightarrow \{s^2\bar{y} - s\,y(0) - y'(0)\} + \bar{y} = \frac{1}{s^2} \quad \text{where } \bar{y} = L\{y(t)\}$$

$$\Rightarrow (s^2\bar{y} - sa - 1) + \bar{y} = \frac{1}{s^2} \quad (\text{Taking } y(0) = a)$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{1}{s^2} + sa + 1$$

$$\Rightarrow \bar{y} = \frac{1}{s^2(s^2+1)} + a\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow y = L^{-1}\{\bar{y}\}$$

$$= L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} + a.L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} \quad \text{--- (1)}$$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ --- (2)

$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$ --- (3)

Dividing (2) by s

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \sin u\,du$$

$$= \left[-\cos u\right]_0^t$$

$$= -(\cos t - \cos 0)$$

$$= -(\cos t - 1)$$

$$= 1 - \cos t.$$

Dividing again by s

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\} = \int_0^t (1-\cos u)\,du$$

$$= \left[u - \sin u\right]_0^t$$

$$= (t-0) - (\sin t - \sin 0)$$

ie $$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t$$

Putting in ①

$$y = (t - \sin t) + a(\cos t) + \sin t$$

$$\Rightarrow y = t + a\cos t \quad \text{——— ④}$$

When $t = \pi$ $\quad y = 0$ (Putting in ④)

$$0 = \pi + a\cos\pi$$

$$\Rightarrow 0 = \pi - a$$

$$\Rightarrow a = \pi$$

Soln is $$y = t + \pi\cos t$$ Ans

Q: Solve $y'' + y = \cos x$ where $y(0) = 0 = y'(0)$.

Soln Given diff. eqn is

$$y'' + y = \cos x$$

Taking Laplace transform.

$$L\{y''\} + L\{y\} = L\{\cos x\}$$

$$\Rightarrow \{s^2\bar{y} - s\,y(0) - y'(0)\} + \bar{y} = \frac{s}{s^2+1}$$

$$\Rightarrow \{s^2\bar{y} - 0 - 0\} + \bar{y} = \frac{s}{s^2+1}.$$

$$\Rightarrow \bar{y}(s^2+1) = \frac{s}{(s^2+1)}$$

$$\Rightarrow \bar{y} = \frac{s}{(s^2+1)^2}.$$

Now, taking inverse Laplace transform

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

ie $\quad y = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ ——— (1)

We have $\quad L\{\sin x\} = \frac{1}{s^2+1}$

$$\Rightarrow L\{x\sin x\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow L\{x\sin x\} = -\left\{\frac{-1}{(s^2+1)^2} \times 2s\right\}$$

$$\Rightarrow L\{x\sin x\} = \frac{2s}{(s^2+1)^2}$$

$$\Rightarrow x\sin x = 2L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}x\sin x.$$

Putting in (1) we get

$$y = \frac{1}{2}x\sin x \quad \text{Ans}$$

Ex Solve: $2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = 1$

Soln Given $\quad 2y'' + 5y' + 2y = e^{-2t}.$

Taking Laplace transform

$$2L\{y''\} + 5L\{y'\} + 2L\{y\} = L\{e^{-2t}\}$$

$$\Rightarrow 2\{s^2\bar{y} - s\,y(0) - y'(0)\} + 5\{s\bar{y} - y(0)\} + 2\bar{y} = \frac{1}{s+2}$$

$$\Rightarrow 2s^2\bar{y} - 2s - 2 + 5s\bar{y} - 5 + 2\bar{y} = \frac{1}{s+2}$$

$$\Rightarrow (2s^2 + 5s + 2)\bar{y} = \frac{1}{s+2} + 2s + 7$$

$$\Rightarrow \bar{y} = \frac{1}{(s+2)(2s^2+5s+2)} + \frac{2s+7}{2s^2+5s+2}$$

$$\Rightarrow \bar{y} = \frac{1}{(s+2)(2s+1)(s+2)} + \frac{2s+7}{(2s+1)(s+2)}$$

$$\Rightarrow \bar{y} = \frac{1}{(s+2)^2(2s+1)} + \frac{2s+7}{(2s+1)(s+2)}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{(s+2)^2(2s+1)}\right\} + L^{-1}\left\{\frac{2s+7}{(2s+1)(s+2)}\right\} \quad \text{--- (1)}$$

Now $L^{-1}\left\{\frac{1}{(s+2)^2(2s+1)}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^2(2s-3)}\right\}$ (Using shifting theorem) (2)

$$L^{-1}\left\{\frac{1}{2s-3}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-3/2}\right\} = \frac{1}{2}e^{\frac{3}{2}t}.$$

Dividing by s

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{2s-3}\right\} = \int_0^t \frac{1}{2}e^{\frac{3}{2}u}\,du = \frac{1}{2}\left[\frac{e^{\frac{3}{2}u}}{\frac{3}{2}}\right]_0^t$$

$$= \frac{1}{3}\left\{e^{\frac{3}{2}t} - 1\right\}$$

Dividing again by s

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{2s-3}\right\} = \int_0^t \frac{1}{3}\left\{e^{\frac{3}{2}u} - 1\right\}du.$$

$$= \frac{1}{3}\left[\frac{e^{\frac{3}{2}u}}{\frac{3}{2}} - u\right]_0^t$$

$$= \frac{2}{9}\left(e^{3/2\,t} - 1\right) - \frac{1}{3}(t - 0)$$

$$= \frac{2}{9}\left(e^{\frac{3}{2}t} - 1\right) - \frac{1}{3}t$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+2)^2(2s+1)}\right\} = e^{-2t}\cdot\left\{\frac{2}{9}\left(e^{\frac{3}{2}t} - 1\right) - \frac{1}{3}t\right\}$$

Also $L^{-1}\left\{\frac{2s+7}{(2s+1)(s+2)}\right\} = L^{-1}\left\{\frac{4}{2s+1} - \frac{1}{s+2}\right\}$

$$= 4L^{-1}\left\{\frac{1}{2s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{4}{2}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 2\cdot e^{-\frac{1}{2}t} - e^{-2t}$$

Putting in (1) we get the sol^n as

$$y = e^{-2t}\left\{\frac{2}{9}\left(e^{\frac{3}{2}t} - 1\right) - \frac{1}{3}t\right\} + 2e^{-\frac{1}{2}t} - e^{-2t}$$

$$y = \frac{2}{9}e^{-\frac{1}{2}t} - \frac{2}{9}e^{-2t} - \frac{1}{3}te^{-2t} + 2e^{-\frac{1}{2}t} - e^{-2t}$$

$$\Rightarrow y = \frac{20}{9}e^{-\frac{1}{2}t} - \frac{11}{9}e^{-2t} - \frac{t}{3}e^{-2t}$$

Ans