Rings

Group A set G along with binary operation "o" is said to be a group if

(i) $a, b \in G \Rightarrow aob \in G$ (Closure prop)

(ii) $ao(boc) = (aob)oc \quad \forall a, b, c \in G$ (Associative prop)

(iii) $\exists$ an element $e \in G$ such that
$$aoe = eoa = a \quad \forall a \in G$$ (Existence of identity)

(iv) For every $a \in G$ $\exists$ $a^{-1} \in G$ such that
$$aoa^{-1} = a^{-1}oa = e$$ (Existence of inverse)

Def^n A group is called an abelian group (commutative group) if $aob = boa \quad \forall a, b \in G$

Def^n A set G with binary operation "o" is said to be a semi-group if

(i) $a, b \in G \Rightarrow aob \in G$ (Closure)

(ii) $ao(boc) = (aob)oc \quad \forall a, b, c \in G$ (Associative)

Notation $ab \longrightarrow aob = a.b$

$a^2 \longrightarrow aoa = a.a = a^2$

$a^3 \longrightarrow aoaoa = a.a.a = a^3$

Generally in a group, we show binary operation by multiplication

Some times in a group we show binary operation by addition. $aoa = a + a = 2a$

$aob = a + b$

$aoaoa = a + a + a = 3a$

Ring: A set R alongwith two binary operations $+$ and $\cdot$ is said to be a ring if

Abelian group under addition:

(i) $a, b \in R \Rightarrow a+b \in R$ $\longrightarrow$ Closure

(ii) $a+(b+c) = (a+b)+c \quad \forall\, a, b, c \in R$ $\longrightarrow$ Associative

(iii) $\exists$ an element $0 \in R$ such that
$$a+0 = 0+a = a \quad \forall\, a \in R \longrightarrow \text{Identity}$$

(iv) For every $a \in R$, $\exists\; -a \in R$ such that
$$a+(-a) = (-a)+a = 0 \longrightarrow \text{Inverse}$$

(v) $a+b = b+a \quad \forall\, a, b \in R$ $\longrightarrow$ Commutative

Semi-group under multiplication:

(vi) For $a, b \in R \Rightarrow a \cdot b \in R$ $\longrightarrow$ Closure

(vii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall\, a, b, c \in R$ $\rightarrow$ Associative

Distributive prop:

$$\left.\begin{array}{ll}\text{(viii)} & a \cdot (b+c) = a \cdot b + ac \\ \text{(ix)} & (b+c) \cdot a = ba + ca\end{array}\right\} \text{ distributive prop.}$$

We denote a ring by $(R, +, \cdot)$

Ring with unity

A ring with multiplicative identity is called ring with unity.

ie If $(R, +, \cdot)$ is a ring and if $\exists$ an element $1 \in R$ such that
$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in R$$
then $(R, +, \cdot)$ is called ring with unity

Commutative ring: A ring $(R, +, \cdot)$ is said to be a commutative ring if

$$a \cdot b = b \cdot a \quad \forall\, a, b \in R$$

---x---

Ex ① The set of real numbers R along with ordinary addition (+) & multiplication (·) is a commutative ring with unity. $(R, +, \cdot)$

② Set of rational numbers Q is also a ring under usual addition & multiplication.

It is also commutative ring with unity

Boolean ring A ring $(R, +, \cdot)$ is said to be a Boolean ring if $\quad a \cdot a = a$

ie $a^2 = a \quad \forall\, a \in R$ (Idempotent prop)

p-ring A ring $(R, +, \cdot)$ is called a p-ring

if $\quad a^p = a$ and $pa = a$

ie $a \cdot a \cdot a \cdots p \text{ times} = a \Rightarrow a^p = a$

& $a + a + a + \cdots p \text{ times} = a \Rightarrow pa = a$

Divisor of Zero (Zero divisor):

An element $a \neq 0$ of a ring R is said to be a divisor of zero if $\exists\, b \neq 0$ in R such that

$$a \cdot b = 0 \text{ or } b \cdot a = 0$$

(Note: b is also a divisor of zero)

Ex Let $(Z_8, +_8, \times_8)$ be a ring

$$Z_8 = \{0,1,2,3,4,5,6,7\}$$

$4 +_8 5 = 1$	$4 \times_8 5 = 4$
$7 +_8 5 = 4$	$7 \times_8 5 = 3$

Here 4 is divisor of zero

$\because \exists\, 2 \in Z_8$ s.t

$$4 \times_8 2 = 0$$

(Here 2 is also divisor of zero)

Ex Set of all second order matrices R with ~~ex~~ integral entries

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in Z \right\}$$

Clearly $(R, +, \cdot)$ is a ring.

Where + is for addition of matrices

$\cdot$ is for multiplication ~~of~~ of matrices

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

Here $A \neq 0 \quad B \neq 0$

but $A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Hence A & B are divisor of zero.