Characteristic of a ring:

The characteristic of a ring is smallest positive integer n such that $na = 0 \ \forall \ a \in R$.

If no such positive integer exists then characteristic of ring is zero.

Ex $Z_6 = \{0, 1, 2, 3, 4, 5\}$

Clearly $(Z_6, +_6, \times_6)$ is a ring.

It's characteristic is 6.

Characteristic of integral Domain/Field

The characteristic of integral domain/Field is smallest positive integer n such that $na = 0$ $\forall \ a \in R$. If no such integer exists then characteristic is 0.

Thm The characteristic of integral domain/Field "R" is 0 or a positive integer n according as the order of unity element (1) of R is of order 0 or n when e is regarded as element of additive group of R.

Proof If $o(1) = 0$

i.e $\exists$ no positive integer n such that

$$n.1 = 0$$

$\Rightarrow$ Characteristic of $R = 0$

If $o(1) = n$ where n is positive integer

$\Rightarrow n.1 = 0$ —— ①

i.e $1+1+1+\cdots$ n-times $= 0$

Let $a \in R$

Now $a+a+a\cdots$ n-times $= a.1 + a.1 + \cdots$ n-ti

$\Rightarrow na = a.1 + a.1 + \cdots$ n times

$\Rightarrow na = a(1+1+1+\cdots$ n times$)$

$\Rightarrow na = a(0)$ (From ①)

$\Rightarrow na = 0$

Hence $na = 0 \; \forall \; a \in R$

Hence char. of $R = n$

Ordered Integral Domain

Let $(R, +, \cdot)$ be any integral domain. It is said to be ordered integral domain if $\exists$ a subset R_+ of R such that.

(i) R_+ is closed w.r.to addition & multiplication

i.e for $a, b \in R_+ \Rightarrow a+b \in R_+$ & $a.b \in R_+$

(ii) For any element $a \in R$ any one of following holds.

either $a = 0$ or $a \in R_+$. or $-a \in R_+$

Elements of R_+ are called positive elements and elements of $R - R_+$ are called negative elements.

Similarly we can define ORDERED FIELD

UNIT: An element $a \in R$ is said to be a unit if multiplicative inverse of a exists in R

Note: Unity (1) is a unit,
but there may be units other than 1.

Idempotent & Nilpotent

An element a in a ring R is said to be idempotent if $a^n = a$ where $a \neq 0$ and nilpotent if $a^n = 0$ when $a \neq 0$ and n is positive integer.

Theorem: If a, b, c are abitrary elements of a ring R, then

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $a \cdot (-b) = -(ab) = (-a) \cdot b$

(iii) $(-a) \cdot (-b) = a \cdot b$

(iv) $a(b-c) = ab - ac$

(v) $(b-c)a = ba - ca$

Do it yourself