Numerical Analysis

Finite Difference

Q Define forward operator and shifting operator and establish the relation between them.

Ans:

Forward operator (Δ) : Increment in x is Δx, then the function will change $f(x)$ to $f(x+h)$ where increment h is constant ie; $\Delta x = h$

$$\Delta f(x) = f(x+h) - f(x)$$

This $\Delta f(x)$ is known as the 1st forward difference of $f(x)$ and the interval h is called the interval of difference.

$$\begin{aligned}\Delta^2 f(x) = \Delta \Delta f(x) &= \Delta\{f(x+h) - f(x)\} \\ &= \Delta f(x+h) - \Delta f(x) \\ &= f(x+2h) - f(x+h) - f(x+h) + f(x) \\ &= f(x+2h) - 2f(x+h) + f(x)\end{aligned}$$

Where $\Delta^2 f(x)$ is called the second difference of $f(x)$ and so on.

Shifting operator (E): The operation of shifting $f(x)$ to $f(x+h)$ is symbolically denoted by $Ef(x)$ ie; $f(x+h) = Ef(x)$.

$$\therefore f(x+2h) = Ef(x+h) = E^2 f(x)$$

In general,

$$f(x+rh) = E^r f(x)$$

Relation between Δ and E

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x) \\ &= Ef(x) - f(x)\end{aligned}$$

$$\therefore Ef(x) = \Delta f(x) + f(x) = (\Delta + 1) f(x)$$

$$\therefore \boxed{E \equiv \Delta + 1}$$

Note: Δ - Delta, $\nabla \to$ Nebla

Backward difference operator: If x has increment h, then ∇ is called Backward difference operator and defined as

$$\nabla f(x) = f(x) - f(x-h)$$
$$\nabla f(x+h) = f(x+h) - f(x) \quad \text{etc.}$$

which are called 1st backward difference

$$\nabla^2 f(x) = \nabla \nabla f(x)$$
$$= \nabla [f(x) - f(x-h)]$$
$$= f(x) - f(x-h) - f(x-h) + f(x-2h)$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

and so on.

—x—

) Prove that the difference of any order can be expressed in terms of the given values of the function alone.

Or

Define nth forward difference.

1:- let $f(x)$ be the function and h be the interval

We have,

$$\Delta f(x) = f(x+h) - f(x)$$
$$\therefore \Delta^2 f(x) = \Delta \Delta f(x)$$
$$= \Delta [f(x+h) - f(x)]$$
$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^3 f(x) = \Delta \cdot \Delta^2 f(x)$$

$$\therefore \Delta^3 f(x) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$
$$= f(x+3h) - f(x+2h) - 2[f(x+2h) - f(x+h)] + f(x+h) - f(x)$$
$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

Proceeding in this way, we get

$$\Delta^n f(x) = f(x+nh) - {}^nC_1 f\{x+(n-1)h\} + {}^nC_2 f\{x+(n-2)h\} - \cdots\cdots + (-1)^n\, {}^nC_n f(x)$$

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<u>Note</u>: If 'a' is constant then $\Delta a = 0$ and $E(a) = a$.

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* <u>Properties of operations Δ and E:</u>

(a) <u>Distributive law</u>: The operations Δ and E obeys distributive law under addition.

ie;

(i) $\Delta\{f(x) + \phi(x)\} = \Delta f(x) + \Delta\phi(x)$

(ii) $E\{f(x) + \phi(x)\} = Ef(x) + E\phi(x)$

where $f(x)$ and $\phi(x)$ are two functions.

<u>Proof</u>:

(i) $L.H.S = \Delta\{f(x) + \phi(x)\}$

$$= \{f(x+h) + \phi(x+h)\} - \{f(x) + \phi(x)\}$$

$$= f(x+h) - f(x) + \phi(x+h) - \phi(x)$$

$$= \Delta f(x) + \Delta\phi(x)$$

$= RHS$

(ii) $L.H.S = E\{f(x) + \phi(x)\}$

$= f(x+h) + \phi(x+h)$

$= Ef(x) + E\phi(x)$

$= R.H.S$

(b) Commutative law: Commutative law is hold for only product of constant and

(i) $\Delta\{cf(x)\} = c\Delta f(x)$

(ii) $E\{cf(x)\} = cEf(x)$

Proof: (i) $L.H.S = cf(x+h) - cf(x)$

$= c\{f(x+h) - f(x)\} = c\Delta f(x) = R.H.S$

(ii) $L.H.S = E\{cf(x)\}$

$= cf(x+h) = cEf(x) = R.H.S$

vvvi for 2 marks

(c) Indices law: Δ and E obey the indices law.

(i) $\Delta^r \Delta^s f(x) = \Delta^{r+s} f(x)$

(ii) $E^r E^s f(x) = E^{r+s} f(x)$

Proof:

(i) $L.H.S = \Delta^r \Delta^s f(x)$

$= (\Delta.\Delta.\Delta \ldots r \text{ times})(\Delta.\Delta.\Delta. \ldots s \text{ times}) f(x)$

$= \{\Delta.\Delta.\Delta. \ldots (r+s) \text{ times}\} f(x)$

$= \Delta^{r+s} f(x) = R.H.S$

(ii) $L.H.S = E^r . E^s f(x)$

$= (E.E.E. \ldots r \text{ times})(E.E.E. \ldots s \text{ times}) f(x)$

$= \{E.E.E. \ldots (r+s) \text{ times}\} f(x) = E^{r+s} f(x) = R.H.S$

② Prove that $E\Delta \equiv \Delta E$. (or Prove that E and Δ are commutative)

Ans:

$$\Delta E f(x) = \Delta f(x+h) = f(x+2h) - f(x+h) \quad \text{—— ①}$$

$$E\Delta f(x) = E\{f(x+h) - f(x)\} = f(x+2h) - f(x+h) \quad \text{—— ②}$$

From ① and ②

$$\Delta E f(x) = E\Delta f(x)$$

$$\therefore E\Delta \equiv \Delta E \quad \text{proved.}$$

—x—

③ Find the value of $E^x y(0)$ {or $E^x f(0)$}

Ans: We know

$$E = 1 + \Delta$$

$$\therefore E^x = (1+\Delta)^x$$

$$= 1 + x\Delta + \frac{x(x-1)}{\underline{|2}}\Delta^2 + \frac{x(x-1)(x-2)}{\underline{|3}}\Delta^3 + \cdots + \Delta^x$$

From above it follows that

$$E^x y(0) = (1+\Delta)^x y(0)$$

$$= \left\{1 + x\Delta + \frac{x(x-1)}{\underline{|2}}\Delta^2 + \frac{x(x-1)(x-2)}{\underline{|3}}\Delta^3 + \cdots + \Delta^x\right\} y(0)$$

$$\therefore E^x y(0) = y(0) + x\Delta y(0) + \frac{x(x-1)}{\underline{|2}}\Delta^2 y(0) + \frac{x(x-1)(x-2)}{\underline{|3}}\Delta^3 y(0) + \cdots + \Delta^x y(0)$$

Ans.

—x—

④ Prove that $E\nabla \equiv \nabla E \equiv \Delta$

Ans:– where h is interval

$$E\nabla f(x) = E\{f(x) - f(x-h)\}$$

$\therefore E\nabla f(x) = Ef(x) - Ef(x-h)$
$= f(x+h) - f(x)$ —— ①

$\nabla E f(x) = \nabla f(x+h)$
$= f(x+h) - f(x)$ —— ②

and $\Delta f(x) = f(x+h) - f(x)$ —— ③
From ①, ② and ③

$$E\nabla f(x) = \nabla E f(x) = \Delta f(x)$$
$$\therefore \quad E\nabla \equiv \nabla E \equiv \Delta \quad \text{proved.}$$

—x—

⑤ If $f(E)$ is a function of E, prove that $f(E)\,a^x = a^x f(a)$.

Ans:-

let $f(E) = b_n E^n + b_{n-1}E^{n-1} + \cdots\cdots + b_0$
where $b_n, b_{n-1}, \cdots, b_0$ are constant
put $E = a$ then
$f(a) = b_n a^n + b_{n-1}a^{n-1} + \cdots\cdots + b_0$

Now. $f(E)\,a^x = (b_n E^n + b_{n-1}E^{n-1} + \cdots + b_0)\,a^x$
$= b_n E^n a^x + b_{n-1}E^{n-1}a^x + \cdots\cdots + b_0 a^x$
$= b_n a^{x+n} + b_{n-1}a^{x+n-1} + \cdots\cdots + b_0 a^x$
$= (b_n a^n + b_{n-1}a^{n-1} + \cdots\cdots + b_0)\,a^x$
$= a^x f(a)$ proved.

—x—

⑥ If $u_x = \sin x$, prove that
$\Delta^2 u_x = cEu_x$, where 'c' is constant.

Ans:-

$\Delta^2 u_x = \Delta^2 \sin x$

$$\Delta^2 u_x = \Delta.\Delta \sin x$$
$$= \Delta\{\sin(x+1) - \sin x\} \quad \text{, Taking } h=1$$
$$= \Delta\{2\cos\frac{2x+1}{2}\sin(\tfrac{1}{2})\}$$
$$= \Delta 2\sin(\tfrac{1}{2})\cos\frac{2x+1}{2} = 2\sin(\tfrac{1}{2})\Delta\cos\frac{2x+1}{2}$$
$$= 2\sin(\tfrac{1}{2})\left\{\cos\left(\frac{2x+3}{2}\right) - \cos\left(\frac{2x+1}{2}\right)\right\}$$
$$= 2\sin(\tfrac{1}{2})\ 2\sin\frac{\frac{2x+3}{2}+\frac{2x+1}{2}}{2}.\sin\frac{\frac{2x+1}{2}-\frac{2x+3}{2}}{2}$$
$$= -4\sin(\tfrac{1}{2})\ \sin\frac{4x+4}{4}\sin(\tfrac{1}{2})$$
$$= -(2\sin\tfrac{1}{2})^2\sin(x+1)$$
$$= cE\sin x \quad \text{where } c = -(2\sin\tfrac{1}{2})^2$$
and $E\sin x = \sin(x+1)$

proved

--x--

(7) Prove that $\Delta u_x v_x = v_x \Delta u_x + u_{x+1}\Delta v_x$

Ans:

$$RHS = v_x \Delta u_x + u_{x+1}\Delta v_x$$
$$= v_x(u_{x+1} - u_x) + u_{x+1}(v_{x+1} - v_x)$$
$$= \cancel{v_x u_{x+1}} - v_x u_x + u_{x+1}v_{x+1} - \cancel{u_{x+1}v_x}$$
$$= u_{x+1}v_{x+1} - v_x u_x$$
$$= \Delta u_x v_x.$$

proved

(8) Prove that $\Delta\left(\frac{u_x}{v_x}\right) = \frac{v_x\Delta u_x - u_x\Delta v_x}{v_x v_{x+1}}$

Ans.