Cancellation law: If R is a ring. and $a, b, c \in R$ where $a \neq 0$ then cancellation law is said to hold

if $ab = ac \Rightarrow b = c$

& $ba = ca \Rightarrow b = c$.

Th^m A ring R is without divisor of zero if & only if the two cancellation laws hold in R.

Proof: Let R is a ring without any divisor of zero. We want to show that the cancellation laws hold in R.

Let $a \neq 0$ & b, c be elements of R and

Let $ab = ac$

$\Rightarrow ab - ac = 0$

$\Rightarrow a(b - c) = 0$

Since R is without divisor of zero

& $a \neq 0 \Rightarrow (b - c) = 0$

ie $b - c = 0$

$\Rightarrow b = c$ (Left cancellation law holds)

Let $ba = ca$

$\Rightarrow ba - ca = 0$

$\Rightarrow (b - c)\,a = 0$

Since R is without divisor of zero

and $a \neq 0 \Rightarrow b - c = 0$

ie $b = c$ (Right cancellation law holds)

(Necessary part is proved)

Sufficient: Let two cancellation laws hold in R
We want to show that R has no divisor of zero
Let us assume that R has a divisor of zero
ie $\exists$ two non zero elements a & b such that

$$ab = 0$$

Now $\quad ab = 0$

$$\Rightarrow a.b = a.0 \quad (\because a.0 = 0)$$

$$\Rightarrow b = 0 \quad \text{(Left cancellation laws)}$$

which is contradiction
Hence R can not have a divisor of zero.

Th^m Prove that a commutative ring D with unity is an integral domain iff $\forall\, a, b, c \in D$ with $a \neq 0$

$$ab = ac \Rightarrow b = c.$$

(D. it)

Th^m Every field is an Integral Domain

Proof A field is a commutative ring with unity and whose non-zero elements are units. (ie inverse of non-zero elements exists)

Now
It is sufficient to show that a field has no divisor of zero.

Let a, b be any two elements of a field F such that $a \cdot b = 0$

Let $a \neq 0$

$\Rightarrow a^{-1}$ exists s.t $a \cdot a^{-1} = a^{-1} \cdot a = 1$

$ab = 0$

$\Rightarrow a^{-1} \cdot (ab) = a^{-1} \cdot 0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1 \cdot b = 0$

$\Rightarrow b = 0$

Similarly if $b \neq 0$ then we will get $a = 0$

Hence $ab = 0$

$\Rightarrow$ Either $a = 0$ or $b = 0$

$\Rightarrow$ F has no divisor of zero

$\Rightarrow$ F is integral domain.

Thm: A skew field has no divisor of zero.

Sol: Let a,b be two elements of a skew field F such that $a \cdot b = 0$

We want to show that either $a = 0$ or $b = 0$

Let $a \neq 0$ $\Rightarrow$ $\exists\, a^{-1} \in F$ such that

$a \cdot a^{-1} = a^{-1} a = 1$.

Now $ab = 0$

$\Rightarrow a^{-1}(ab) = a^{-1} \cdot 0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1 \cdot b = 0$

$\Rightarrow b = 0$

So Either $a = 0$ or $b = 0$

Hence F has no divisor of zero.

Thm: Every finite integral domain is a field.

Soln Let $(D, +, \cdot)$ be a finite integral domain

Let $a \neq 0$ be any element of D.

Let $D = \{x_1, x_2, x_3, \dots x_n\}$

Let $D' = \{ax_1, ax_2, ax_3 \dots ax_n\}$

Clearly all the elements of D' are distinct

If not Let $ax_i = ax_j$

$\Rightarrow x_i = x_j \quad (\because a \neq 0)$

Since elements of D' are elements of D (By closure prop.)

$\Rightarrow \exists$ some $x_j \in D$ such that

$ax_j = 1 \quad (\because 1 \in D)$

$\Rightarrow a \cdot x_j = x_j \cdot a = 1 \quad (\because$ D is commutative$)$

$\Rightarrow x_j = a^{-1}$

So multiplicative inverse of a exists

$\Rightarrow$ D is a field.