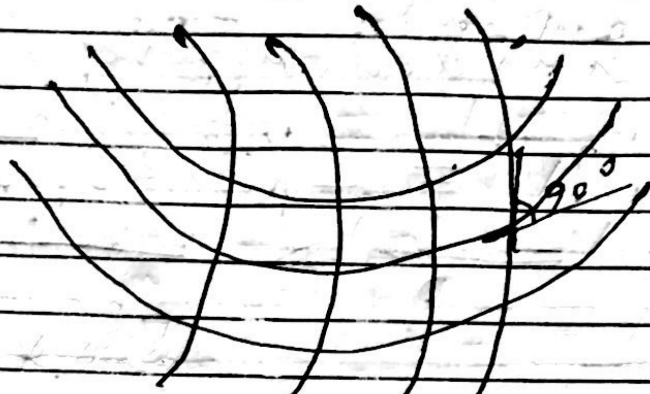
Santosh 22/4/2020

ORTHOGONAL TRAJECTORIES

UG Sem III
Paper- CC

When one family of curve is orthogonal to another family of curve. Then the curves are called orthogonal. The path of orthogonal family of curves is called orthogonal trajectories.



Working rule: (i) If the family of curve is in cartesian form.

Step1 Differentiate Eq. of curve w.r.t x

Step2 Eliminate arbitrary constant "c" from the given equation and the relation obtained after diffing the Eq. of curve.

Step3 For finding O.T., replace $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}} = -\frac{dx}{dy}$.

Step4 After seperation variable, then integrate we will get the Eq. of O.T.

Now we are going to solve some Problems.

Problem ① Find the O.T. of the curve

$$x^2+y^2-ay=0$$

Ans Given Equation of the curve is

$$x^2+y^2-ay=0 \quad \text{——— ①}$$

Here "a" is the parametric constant

Differentiating ① w.r.t. x

$$2x+2y\frac{dy}{dx}-a\frac{dy}{dx}=0$$

$$\Rightarrow 2x+2y\frac{dy}{dx}=a\frac{dy}{dx}$$

$$\Rightarrow a=2x+2y\frac{dy}{dx} \Big/ \frac{dy}{dx}$$

Eliminating "a" from Eq ①, we have

$$x^2+y^2-\frac{2x+2y\frac{dy}{dx}}{\frac{dy}{dx}}\cdot y=0$$

$$\Rightarrow (x^2+y^2)\frac{dy}{dx}-\left(2x+2y\frac{dy}{dx}\right)\cdot y=0$$

$$\Rightarrow (x^2+y^2)\frac{dy}{dx}-2xy-2y^2\frac{dy}{dx}=0$$

$$\Rightarrow (x^2+y^2-2y^2)\frac{dy}{dx}=2xy$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

For finding Equation of O.T. replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{2xy}{x^2-y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$, It is a homogeneous Eq.

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v} = \frac{-v^2-1}{2v}$$

$$\Rightarrow \frac{2v}{-v^2-1}\,dv = \frac{dx}{x}$$

$$\Rightarrow -\int \frac{2v}{v^2+1}\,dv = \int \frac{dx}{x}$$

$\Rightarrow \log c - \log|v^2+1| = \log x$, where c is constant

$$\Rightarrow \log \frac{c}{v^2+1} = \log x$$

$$\Rightarrow \frac{c}{v^2+1} = x \Rightarrow c = x(v^2-1)$$

$$\Rightarrow x\left(\frac{y^2}{x^2}+1\right) = c \Rightarrow \frac{y^2+x^2}{x} = c$$

$\Rightarrow y^2 + x^2 = cx$. This the required sol^n.

Problem ② Find the O.T. of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1, \ \lambda \text{ being a parameter}$$

Ans: Given Equation of curve is $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ — ①

Differentiating ① w.r.t. x

$$\frac{2x}{a^2} + \frac{2y\frac{dy}{dx}}{a^2+\lambda} = 0$$

$$\Rightarrow \frac{y\frac{dy}{dx}}{a^2+\lambda} = -\frac{x}{a^2}$$

$$\Rightarrow \frac{1}{a^2+\lambda} = \frac{-\frac{x}{a^2}}{y\frac{dy}{dx}} \quad \text{— ②}$$

Eliminating λ from Eq ① & ② we have

$$\frac{x^2}{a^2} + y^2\left(\frac{-x}{a^2 / y\frac{dy}{dx}}\right) = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{xy^2}{a^2 y\frac{dy}{dx}} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - 1 = \frac{xy^2}{a^2 y\frac{dy}{dx}}$$

$\Rightarrow$ For O.T. replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\frac{x^2-a^2}{a^2} = \frac{xy}{a^2\left(-\frac{dx}{dy}\right)}$$

$\Rightarrow \quad x^2 - a^2 = -xy\frac{dy}{dx}$

$\Rightarrow \quad \frac{x^2-a^2}{x}dx = -y\,dy$

$\Rightarrow \quad \int\left[x - \frac{a^2}{x}\right]dx = -\int y\,dy$

$\Rightarrow \quad \frac{x^2}{2} - a^2\log x = -\frac{y^2}{2} + c$

$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} - a^2\log x = c$

$\Rightarrow \quad x^2 + y^2 - 2a^2\log x = 2c$

$\Rightarrow \quad x^2 + y^2 - 2a^2\log x = C$, where $C = 2c$

This is the required Eq. of O.T.

<u>Working rule (ii)</u> If the family of curve is in polar form: i.e $r = f(\theta)$

<u>Step 1</u>: Differentiate the given curve and eliminate the arbitrary constant.

Step 2: replace $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$

Step 3: Integrate the equation, we will get the required Eq. of O.T.

Problem (3) Find the O.T. of $r^n = a^n \cos n\theta$.

Ans: Given Eq. of curve is $r^n = a^n \cos n\theta$

Taking log on both sides, we have

$\log r^n = \log(a^n \cdot \cos n\theta)$

$\Rightarrow n \log r = \log a^n + \log(\cos n\theta)$

Now diff$\underline{\text{ing}}$ w.r.t θ.

$\Rightarrow n \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \cdot -n \sin n\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$

$\Rightarrow r\frac{d\theta}{dr} = -\cot n\theta$

For finding O.T. replacing $r\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$

ie $\cancel{-}\frac{1}{r}\frac{dr}{d\theta} = \cancel{-}\cot n\theta$.

$\Rightarrow \frac{dr}{r} = \cot n\theta \, d\theta$

$\Rightarrow \int \frac{dr}{r} = \int \cot n\theta \, d\theta$

$\Rightarrow \log r = \frac{\log(\sin n\theta)}{n} + \log c$

$\Rightarrow n \log r = \log(\sin n\theta) + n \log c$

$\Rightarrow \log r^n = \log(c^n \cdot \sin n\theta) \Rightarrow r^n = c^n \sin n\theta$ Ans