

★ Another type Problem:

1) Given that $y_0 + y_8 = 80$, $y_1 + y_7 = 10$, $y_2 + y_6 = 5$, $y_3 + y_5 = 10$, find y_4 .

Ans: Since 8 entries are given and so it is 7th degree Polynomial.

$$\therefore \Delta^8 y_0 = 0$$

$$\Rightarrow (E-1)^8 y_0 = 0$$

$$\Rightarrow (E^8 - {}^8C_1 E^7 + {}^8C_2 E^6 - {}^8C_3 E^5 + {}^8C_4 E^4 - {}^8C_5 E^3 + {}^8C_6 E^2 - {}^8C_7 E + 1) y_0 = 0$$

$$\Rightarrow E^8 y_0 - {}^8C_1 E^7 y_0 + {}^8C_2 E^6 y_0 - {}^8C_3 E^5 y_0 + {}^8C_4 E^4 y_0 - {}^8C_5 E^3 y_0 + {}^8C_6 E^2 y_0 - {}^8C_7 E y_0 + y_0 = 0$$

$$\Rightarrow y_8 - 8y_7 + 28y_6 - 56y_5 + 70y_4 - 56y_3 + 28y_2 - 8y_1 + y_0 = 0$$

{ $\because mC_{m-r} = mC_r$ }

$$\Rightarrow (y_8 + y_0) - 8(y_7 + y_1) + 28(y_6 + y_2) - 56(y_5 + y_3) + 70y_4 = 0$$

$$\Rightarrow 80 - 8 \times 10 + 28 \times 5 - 56 \times 10 + 70y_4 = 0$$

$$\Rightarrow 140 - 560 + 70y_4 = 0$$

$$\Rightarrow -420 + 70y_4 = 0$$

$$\Rightarrow 70y_4 = 420$$

$$\therefore \boxed{y_4 = 6}$$

2) If U_x is Polynomial of 7th degree in x and
 $U_0 + U_8 = 1.924$, $U_1 + U_7 = 1.959$, $U_2 + U_6 = 1.982$,
 $U_3 + U_5 = 1.996$, find U_4 .

Ans: we have 7th degree Polynomial.

$$\therefore \Delta^8 y_0 = 0$$

$$\Rightarrow (E+1)^8 y_0 = 0$$

$$\Rightarrow (E^8 - 8C_1 E^7 + 8C_2 E^6 - 8C_3 E^5 + 8C_4 E^4 - 8C_5 E^3 + 8C_6 E^2 - 8C_7 E + 1) y_0 = 0$$

$$\Rightarrow E^8 y_0 - 8C_1 E^7 y_0 + 8C_2 E^6 y_0 - 8C_3 E^5 y_0 + 8C_4 E^4 y_0 - 8C_5 E^3 y_0 + 8C_6 E^2 y_0 - 8C_7 E y_0 + y_0 = 0$$

$$\Rightarrow y_8 - 8y_7 + 28y_6 - 56y_5 + 70y_4 - 56y_3 + 28y_2 - 8y_1 + y_0 = 0$$

$$\Rightarrow (y_8 + y_0) - 8(y_7 + y_1) + 28(y_6 + y_2) - 56(y_5 + y_3) + 70y_4 = 0$$

$$\Rightarrow 1.924 - 8 \times 1.959 + 28 \times 1.982 - 56 \times 1.996 + 70y_4 = 0$$

$$\Rightarrow 1.924 - 15.672 + 55.496 - 111.776 + 70y_4 = 0$$

$$\Rightarrow 57.42 - 127.448 + 70y_4 = 0$$

$$\Rightarrow -70.028 + 70y_4 = 0$$

$$\Rightarrow 70y_4 = 70.028$$

$$\therefore \boxed{y_4 = 1.0004}$$

1) Prove that

$$\Delta^n u_x = u_{x+m} - {}^m C_1 u_{x+m-1} + {}^m C_2 u_{x+m-2} + \dots + (-1)^m u_x$$

$$\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + \dots + (-1)^n u_x$$

Ans: Proof:

$$\begin{aligned} \text{L.H.S} \quad \Delta^n u_n &= (E-1)^n u_n \\ &= \{ E^n - nC_1 E^{n-1} + nC_2 E^{n-2} - \dots + (-1)^n \} u_n \end{aligned}$$

$$= E^m U_x - m E^{m-1} U_x + m(m-1) E^{m-2} U_x + \dots + (-1)^n U_x$$

$$= U_{x+m} - {}^m C_1 U_{x+m-1} + {}^m C_2 U_{x+m-2} + \dots + (-1)^m U_x$$

Proved

2) Prove that

$$y_x = y_m - {}^{m-x}C_1 \Delta y_{m-1} + {}^{m-x}C_2 \Delta^2 y_{m-2} - \dots + (-1)^{m-x} \Delta^{m-x} y_{m-(m-x)}$$

Ans: Proof:

$$\text{R.H.S } y_m - {}^{m-x}C_1 \Delta y_{m-1} + {}^{m-x}C_2 \Delta^2 y_{m-2} - \dots + (-1)^{m-x} \Delta^{m-x} y_{m-(m-x)}$$

$$= y_m - n-x C_1 \Delta E^{-1} y_m + n-x C_2 \Delta^2 E^{-2} y_m - \dots$$

$$= \left\{ 1 - {}^{n-x}C_1 \Delta E^{-1} + {}^{n-x}C_2 \Delta^2 E^{-2} - \dots + (-1)^{n-x} \Delta^{n-x} E^{-(n-x)} \right\} \cdot$$

$$= \left\{ 1 - \frac{\Delta}{E} + \frac{\Delta^2}{E^2} - \dots + (-1)^{m-x} \frac{\Delta^{m-x}}{E^{m-x}} \right\} y_m$$

$$= \left(1 - \frac{\Delta}{E} \right)^{m-x} y_m$$

$$= \frac{(E - \Delta)^{m-x}}{E^{m-x}} y_m$$

$$= \frac{1}{E^{m-x}} y_m \quad \{ \because 1 + \Delta = E \}$$

$$= E^{-m+x} y_m$$

$$= y_{m-m+x}$$

$$= y_x = \text{L.H.S}$$

proved

3) Prove that

$$y_k = \sum_{i=0}^k {}^k C_i \Delta^i y_0$$

Ans. R.H.S. $\sum_{i=0}^k {}^k C_i \Delta^i y_0$

$$= {}^k C_0 \Delta^0 y_0 + {}^k C_1 \Delta y_0 + {}^k C_2 \Delta^2 y_0 + \dots + {}^k C_k \Delta^k y_0$$

$$= y_0 + {}^k C_1 \Delta y_0 + {}^k C_2 \Delta^2 y_0 + \dots + {}^k C_k \Delta^k y_0$$

$$= (1 + {}^k C_1 \Delta + {}^k C_2 \Delta^2 + \dots + {}^k C_k \Delta^k) y_0$$

$$= (1 + \Delta)^k y_0$$

$$= E^k y_0$$

$$= y_{0+k} = y_k = \text{L.H.S}$$

proved

4) Prove that

$$U_0 + \frac{U_1 x}{1!} + \frac{U_2 x^2}{2!} + \frac{U_3 x^3}{3!} + \dots = e^x (U_0 + x \Delta U_0 + \frac{x^2 \Delta^2 U_0}{2!} + \dots)$$

Ans: L.H.S

$$U_0 + \frac{U_1 x}{1!} + \frac{U_2 x^2}{2!} + \frac{U_3 x^3}{3!} + \dots + \infty$$

$$= U_0 + \frac{E U_0 x}{1!} + \frac{E^2 U_0 x^2}{2!} + \frac{E^3 U_0 x^3}{3!} + \dots$$

$$= \left(1 + \frac{E x}{1!} + \frac{E^2 x^2}{2!} + \frac{E^3 x^3}{3!} + \dots \right) U_0$$

$$= e^{E x} U_0$$

$$= e^{(1+\Delta)x} U_0$$

$$= (e^x \cdot e^{\Delta x}) U_0$$

$$= e^x \left(1 + \frac{\Delta x}{1!} + \frac{\Delta^2 x^2}{2!} + \dots \right) U_0$$

$$= e^x \left(U_0 + x \Delta U_0 + \frac{x^2 \Delta^2 U_0}{2!} + \dots \right) = R.H.S.$$

Proved

5) Prove that

$$U_0 + U_1 + U_2 + \dots + U_m = {}^{m+1}C_1 U_0 + {}^{m+1}C_2 \Delta U_0 + \dots + \Delta^m U_0$$

Ans: L.H.S

$$U_0 + U_1 + U_2 + \dots + U_m$$

$$= U_0 + E U_0 + E^2 U_0 + \dots + E^m U_0$$

$$= (1 + E + E^2 + \dots + E^m) U_0$$

$$\begin{aligned}
 &= \frac{1 - E^{m+1}}{1 - E} U_0 \\
 &= \frac{E^{m+1} - 1}{E - 1} U_0 \\
 &= \frac{(1 + \Delta)^{m+1} - 1}{\Delta} U_0 \\
 &= \frac{1}{\Delta} [1 + {}^{m+1}C_1 \Delta + {}^{m+1}C_2 \Delta^2 + \dots + {}^{m+1}C_{m+1} \Delta^{m+1}] U_0 \\
 &= [{}^{m+1}C_1 + {}^{m+1}C_2 \Delta + \dots + \Delta^m] U_0 \\
 &= {}^{m+1}C_1 U_0 + {}^{m+1}C_2 \Delta U_0 + \dots + \Delta^m U_0 = R.H.S
 \end{aligned}$$

proved

6) Prove that

$$\Delta^m U_{x-m} = U_x - {}^mC_1 U_{x-1} + {}^mC_2 U_{x-2} - {}^mC_3 U_{x-3} + \dots + (-1)^m {}^mC_m U_{x-m}$$

Ans: R.H.S $U_x - {}^mC_1 U_{x-1} + {}^mC_2 U_{x-2} - {}^mC_3 U_{x-3} + \dots + (-1)^m {}^mC_m U_{x-m}$

$$= U_x - {}^mC_1 E^{-1} U_x + {}^mC_2 E^{-2} U_x - {}^mC_3 E^{-3} U_x + \dots + (-1)^m E^{-m} U_x$$

$$= \{1 - {}^mC_1 E^{-1} + {}^mC_2 E^{-2} - {}^mC_3 E^{-3} + \dots + (-1)^m E^{-m}\} U_x$$

$$= \left\{1 - {}^mC_1 \frac{1}{E} + {}^mC_2 \frac{1}{E^2} - {}^mC_3 \frac{1}{E^3} + \dots + (-1)^m \frac{1}{E^m}\right\} U_x$$

$$= \left(1 - \frac{1}{E}\right)^n U_n$$

$$= \left(\frac{E-1}{E}\right)^n U_n$$

$$= \left(\frac{\Delta}{E}\right)^n U_n$$

$$= \Delta^n E^{-n} U_n$$

$$= \Delta^n U_{n-m} = \text{L.H.S.} \quad \text{proved}$$

7) Prove that

$$U_{n+m} = U_n + {}^nC_1 \Delta U_{n-1} + {}^{n+1}C_2 \Delta^2 U_{n-2} + \dots + \Delta^n U_0$$

Ans: R.H.S.

$$U_n + {}^nC_1 \Delta U_{n-1} + {}^{n+1}C_2 \Delta^2 U_{n-2} + \dots + \Delta^n U_0$$

$$= U_n + {}^nC_1 \Delta U_{n-1} + {}^{n+1}C_2 \Delta^2 U_{n-2} + \dots + \Delta^n U_{n-m}$$

$$= U_n + {}^nC_1 \Delta E^{-1} U_n + {}^{n+1}C_2 \Delta^2 E^{-2} U_n + \dots + \Delta^n E^{-n} U_n$$

$$= \left(1 + {}^nC_1 \Delta E^{-1} + {}^{n+1}C_2 \Delta^2 E^{-2} + \dots + \Delta^n E^{-n}\right) U_n$$

$$= (1 - \Delta E^{-1})^{-n} U_n$$

$$= \left(1 - \frac{\Delta}{E}\right)^{-n} U_n$$

$$= \left(\frac{E-\Delta}{E}\right)^{-n} U_n$$

$$= \left(\frac{1}{E}\right)^{-n} U_n$$

$$= E^n U_n = U_{n+n} = \text{L.H.S.} \quad \text{proved}$$

8) Prove that

$$Ux = Ux-1 + \Delta Ux-2 + \Delta^2 Ux-3 + \dots + \Delta^{m-1} Ux-m + \Delta^m Ux-m$$

Ans: R.H.S

$$Ux-1 + \Delta Ux-2 + \Delta^2 Ux-3 + \dots + \Delta^{m-1} Ux-m + \Delta^m Ux-m$$

$$= Ux-1 + \Delta E^{-1} Ux-1 + \Delta^2 E^{-2} Ux-1 + \dots + \Delta^{m-1} E^{-(m-1)} Ux-1 + \Delta^m Ux-m$$

$$= \{1 + \Delta E^{-1} + \Delta^2 E^{-2} + \dots + \Delta^{m-1} E^{-(m-1)}\} Ux-1 + \Delta^m Ux-m$$

$$= \frac{1 - (\Delta E^{-1})^m}{1 - \Delta E^{-1}} Ux-1 + \Delta^m Ux-m$$

$$= \frac{1 - \Delta^m E^{-m}}{1 - \Delta E^{-1}} Ux-1 + \Delta^m Ux-m$$

$$= \frac{1 - \Delta^m E^{-m}}{E - \Delta} E^{-1} Ux + \Delta^m Ux-m$$

$$= (1 - \Delta^m E^{-m}) E^{-1} E Ux + \Delta^m Ux-m$$

$$= (1 - \Delta^m E^{-m}) Ux + \Delta^m E^{-m} Ux$$

$$= Ux - \cancel{\Delta^m E^{-m} Ux} + \cancel{\Delta^m E^{-m} Ux}$$

$$= Ux = \text{L.H.S} \quad \text{proved}$$

9)

Prove that

$$\begin{aligned} \kappa U_1 + \kappa^2 U_2 + \kappa^3 U_3 + \dots \infty &= \frac{\kappa}{1-\kappa} U_1 + \frac{\kappa^2}{(1-\kappa)^2} \Delta U_1 \\ &+ \frac{\kappa^3}{(1-\kappa)^3} \Delta^2 U_1 + \dots \infty \end{aligned}$$

Ans.

R.H.S $\frac{\kappa}{1-\kappa} U_1 + \frac{\kappa^2}{(1-\kappa)^2} \Delta U_1 + \frac{\kappa^3}{(1-\kappa)^3} \Delta^2 U_1 + \dots \infty$

$$= \frac{\kappa}{1-\kappa} \left\{ 1 + \frac{\kappa}{1-\kappa} \Delta + \frac{\kappa^2}{(1-\kappa)^2} \Delta^2 + \dots \right\} U_1$$

$$= \frac{\kappa}{1-\kappa} \left\{ \frac{1 - \kappa \Delta}{1-\kappa} \right\}^{-1} U_1$$

$$= \frac{\kappa}{1-\kappa} \left\{ \frac{1 - \kappa - \kappa \Delta}{1-\kappa} \right\}^{-1} U_1$$

$$= \frac{\kappa}{1-\kappa} \left[\frac{1 - \kappa(1 + \Delta)}{1-\kappa} \right]^{-1} U_1$$

$$= \frac{\kappa}{1-\kappa} \left[\frac{1 - \kappa E}{1-\kappa} \right]^{-1} U_1$$

$$= \frac{\kappa}{1-\kappa} \left(\frac{1-\kappa}{1-E\kappa} \right) U_1$$

$$= \frac{\kappa}{1-E\kappa} U_1$$

$$= \frac{\kappa E}{1-E\kappa} U_0$$

$$= \kappa E (1-E\kappa)^{-1} U_0$$

$$= \kappa E (1 + E\kappa + E^2 \kappa^2 + E^3 \kappa^3 + \dots \infty) U_0$$

$$= \kappa E U_0 + \kappa^2 E^2 U_0 + \kappa^3 E^3 U_0 + \dots \infty$$

$$= \kappa U_1 + \kappa^2 U_2 + \kappa^3 U_3 + \dots \infty = \text{L.H.S} \quad \text{Proved}$$

10) Prove that
 $\Delta U_x + \Delta U_{x+1} + \Delta U_{x+2} + \dots + \Delta U_{x+m-1} = U_{x+m} - U_x$

Ans: L.H.S

$$\begin{aligned}
 & \Delta U_x + \Delta U_{x+1} + \Delta U_{x+2} + \dots + \Delta U_{x+m-1} \\
 &= \Delta (U_x + E U_x + E^2 U_x + \dots + E^{m-1} U_x) \\
 &= \Delta (1 + E + E^2 + \dots + E^{m-1}) U_x \\
 &= \Delta \frac{E^m - 1}{E - 1} U_x \\
 &= \Delta \frac{E^m - 1}{\Delta} U_x \\
 &= (E^m - 1) U_x \\
 &= U_{x+m} - U_x = \text{R.H.S} \quad \text{Proved}
 \end{aligned}$$

11) Prove that
 $U_x + 2\Delta U_x + 3\Delta^2 U_x + \dots = \frac{1}{4} \left[U_x + U_{x+1} + \frac{3}{4} U_{x+2} + \frac{1}{2} U_{x+3} + \frac{5}{16} U_{x+4} + \dots \right]$

Ans: L.H.S

$$\begin{aligned}
 & U_x + 2\Delta U_x + 3\Delta^2 U_x + \dots \\
 &= [1 + 2\Delta + 3\Delta^2 + \dots] U_x \\
 &= (1 - \Delta)^{-2} U_x \\
 &= (1 - E + 1)^{-2} U_x \\
 &= (2 - E)^{-2} U_x
 \end{aligned}$$

$$= 2^{-2} \left(1 - \frac{E}{2}\right)^{-2} U_n$$

$$= \frac{1}{4} \left[1 + 2 \frac{E}{2} + 3 \frac{E^2}{4} + 4 \frac{E^3}{8} + 5 \frac{E^4}{16} + \dots \right] U_n$$

$$= \frac{1}{4} \left[U_n + U_{n+1} + \frac{3}{4} U_{n+2} + \frac{1}{2} U_{n+3} + \frac{5}{16} U_{n+4} + \dots \right]$$

= R.H.S. proved

Q. Define functions $x^{(m)}$ and $x^{(-m)}$ obtained their n^{th} difference. Distinguishing between the cases of them.

Ans: (i) The factorial functions defined as

$$x^{(m)} = x(x-1)(x-2) \dots (x-m+1), \quad h=1$$

$$\Delta x^{(m)} = (x+1)^{(m)} - x^{(m)}$$

$$= (x+1)x(x-1)(x-2) \dots \{x+1-(m-1)\} - \{x(x-1)(x-2) \dots (x-m+1)\}$$

$$= (x+1)x(x-1)(x-2) \dots (x-m+2) - \{x(x-1)(x-2) \dots (x-m+1)\}$$

$$= x(x-1)(x-2) \dots (x-m+2) \{x+1 - x + m - 1\}$$

$$= m x(x-1)(x-2) \dots (x-m+2)$$

$$= m x^{(m-1)}$$