Convergency and Divergency of series

1) If the Sum of infinite Series is finite and unique then series is called convergent series.

2) If the Sum of infinite series is infinite then series is called divergent series.

3) Oscillatory Series :- If the Sum of infinite Series is finite and non-unique then Series is called oscillatory series.

NOTE :-

① $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots\ldots + u_{n-1} + u_n + u_{n+1} + \ldots$ to ∞

② $\sum_{n=1}^{\infty} u_n$ be series of +ve term (all term are positive)

REMARK :-

① $\sum_{n=1}^{\infty} u_n <$ finite

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent

② $\sum_{n=1}^{\infty} u_n >$ infinite

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent

Never do

1) $\sum_{n=1}^{\infty} u_n >$ finite

noting to say

2) $\sum_{n=1}^{\infty} u_n <$ infinite

Article :— ① G.P test

Statement :— The G.P series is convergent if c.r < 1, divergent if c.r ≥ 1 and oscillatory if c.r = −1

Ans:- Let G.P be $a, ar, ar^2, \ldots\ldots$

Let S_n = Sum of n term

$$S_n = \frac{a(1-r^n)}{(1-r)} \text{ if } r < 1$$

$$= \frac{a(r^n-1)}{(r-1)} \text{ if } r > 1$$

$$= na \text{ if } r = 1$$

Case I :— Let $r < 1$

S_∞ = Sum of infinite term of G.P

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a(1-0)}{1-r} \quad \left[\because r < 1,\ \lim_{n\to\infty} r^n = 0\right]$$

$$= \frac{a}{1-r}$$

∴ G.P is convergent

Case (II) :— Let $r > 1$

S_∞ = Sum of infinite term of G.P

$= \lim_{n \to \infty} S_n$

$= \lim_{n \to \infty} \frac{a(r^n - 1)}{r - 1}$

$= \frac{a(\infty - 1)}{r - 1}$ $\left[\begin{array}{l} \because r > 1 \\ \lim_{n \to \infty} r^n = \infty \end{array} \right]$

= infinite

$\therefore$ G.P is divergent.

Case (III) :— Let $r = 1$

S_∞ = Sum of infinite term of G.P

$= \lim_{n \to \infty} S_n$

$= \lim_{n \to \infty} na$

= infinite

$\therefore$ G.P is divergent

Case (IV) :- Let $r = -1$

$$S_n = a + ar + ar^2 + ar^3 + \ldots\ldots$$

$$= a - a + a - a + \ldots\ldots\ldots \text{ to } n \text{ term}$$

$$= \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$S_\infty = \lim_{n\to\infty} S_n$$

$$= a \text{ or } 0$$

$\therefore$ G.P is oscillatory

Auxilliary Series

The Series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is called auxilliary Series

NOTE :- Infinite A.P is always divergent

2. Article :- State and prove auxilliary Series test.

or

Discuss the convergency of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Ans :- Statement :- The auxilliary series is convergent if $p>1$ and divergent if $p \leq 1$

Proof :- Case (I) - Let $p>1$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots\cdots \text{(next 8 term)}$$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots\cdots \text{(next 8 term)}$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots\cdots$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots\cdots$$

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$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \cdots$ infinite

= Sum of infinite G.P of c.r $= \frac{1}{2^{p-1}} < 1$ as $p > 1$

So it, is convergent

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is finite}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent

<u>Case (II)</u> :- Let $p = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + (\text{next } 8 \text{ term})$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + (\text{next } 8 \text{ term})$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

= 1 + (Sum of infinite G.P of c.r = 1)

~~So, it is divergent~~

$\sum_{n=1}^{\infty} \frac{1}{n^p} >$ infinite

$\therefore$ it is divergent.

Case(III) :- Let $p < 1$

$$\frac{1}{2^p} > \frac{1}{2}, \quad \frac{1}{3^p} > \frac{1}{3}, \quad \frac{1}{4^p} > \frac{1}{4}, \ldots$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Simillary as by Case(II) the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent hence the statement proved.

Alternating Series: –

$$u_1 - u_2 + u_3 - u_4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

Article-3 :- State and prove Leibnitz's test
or State and prove alternating Series test

Statement :- The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ is convergent iff $\lim_{n\to\infty} u_n = 0$ and $|u_{n-1}| > |u_n|$

Proof :- $S = \sum_{n=1}^{\infty}(-1)^{n+1}u_n = (u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\ldots\ldots$

$= (\text{+ve quantity}) + (\text{+ve quantity}) + (\text{+ve quantity}) + \ldots$ $\{\because |u_{n-1}| > |u_n|\}$

$= a\ (\text{+ve})\ \text{quantity} \quad \text{—— (1)}$

Again

$S = u_1 - \{(u_2-u_3)+(u_4-u_5)+\ldots\ldots\}$

$= u_1 - \{(\text{+ve quantity}) + (\text{+ve quantity}) + \ldots\}$

$= u_1 - \{a\ (\text{+ve})\ \text{quantity}\} \quad \text{—— (2)}$

From (1) and (2)

$u_1 > (u_2-u_3)+(u_4-u_5)+\ldots\ldots$

$(u_2-u_3)+(u_4-u_5)+\ldots\ldots < u_1$

$u_2-u_3+u_4-u_5+\ldots\ldots\ldots < \text{finite}$

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$\Rightarrow u_1 - u_2 + u_3 - u_4 + \ldots\ldots < \text{finite}$ —— (3)

Now, we shall prove that limiting sum for even terms and odd term are equal.

Let S_{2n+1} = Sum of $(2n+1)$ term

$$S_{2n+1} = u_1 - u_2 + u_3 - u_4 + \ldots\ldots - u_{2n} + u_{2n+1}$$

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} \quad \left[\because \lim_{n\to\infty} u_n = 0,\ \lim_{n\to\infty} u_{2n+1} = 0\right] \text{—— (4)}$$

From (3) and (4) we say

$u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ is finite and unique and so, it is convergent.

~~Rember~~ Remember :—

(1) $\sum_{n=m}^{\alpha} u_n = u_m + u_{m+1} + u_{m+2} + \dots$

(2) If $\sum_{n=m}^{\alpha} u_n$ is convergent

$\sum_{n=1}^{\alpha} u_n$ is convergent

(3) If $\sum_{n=m}^{\alpha} u_n$ is divergent

$\sum_{n=1}^{\alpha} u_n$ is divergent

Defination of Modulus

(1) $|x| = a$

$x = \pm a$

(2) $|x| < a$

$-a < x < a$

(3) $x \in]-a, a[$

(4) $|x| \leq a$

$-a \leq x \leq a$

(5) $|x-a| \leq l$

$\Rightarrow -l \leq x-a \leq l$

$\Rightarrow a-l \leq x \leq l+a$

Definition of infinite limit:—

$\exists$ - There exist
$\forall$ - For all

$$\lim_{n\to\infty} u_n = l$$

i.e for given ϵ (very small), $\exists$ a large +ve integer m such that

$$|u_n - l| < \epsilon \quad \forall\ n \geq m$$

Article-4 State and prove Comparison test.

Statement:— If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of +ve terms then both the series converge or diverge together if

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{non-zero finite}$$

Proof:— Let $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (say), where k is non-zero finite

i.e for very small $\epsilon > 0$ $\exists$ a large +ve integer m such that

$$\left|\frac{u_n}{v_n} - k\right| < \epsilon \quad \forall\ n \geq m$$

$$\Rightarrow -\epsilon < \frac{u_n}{v_n} - k < \epsilon \quad \forall\ n \geq m$$

$$\Rightarrow k - \epsilon < \frac{u_n}{v_n} < k + \epsilon \quad \forall\ n \geq m \quad \text{———} \ (1)$$

Case (I):– Let $\sum_{n=1}^{\infty} v_n$ be convergent

from (1)

$$\frac{u_n}{v_n} < K+\epsilon \quad \forall\, n \geq m$$

$$u_n < (K+\epsilon)v_n \quad \forall\, n \geq m$$

$\therefore\; u_m < (K+\epsilon)v_m$

$u_{m+1} < (K+\epsilon)v_{m+1}$

$u_{m+2} < (K+\epsilon)v_{m+2}$

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Adding all

$$\sum_{n=m}^{\infty} u_n < (K+\epsilon)\sum_{n=m}^{\infty} v_n$$

$\therefore \sum_{n=m}^{\infty} u_n <$ finite

$\Rightarrow \sum_{n=m}^{\infty} u_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent

Case (II):– Let $\sum_{n=1}^{\infty} v_n$ be divergent

from (1)

$$\frac{u_n}{v_n} > K-\epsilon \quad \forall\, n \geq m$$

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$u_n > (k-\epsilon) v_n \quad \forall \; n \geq m$

$\therefore u_m > (k-\epsilon) v_m$

$u_{m+1} > (k-\epsilon) v_{m+1}$

$u_{m+2} > (k-\epsilon) v_{m+2}$

$\ldots \ldots \ldots$

Adding all

$$\sum_{n=m}^{\alpha} u_n > (k-\epsilon) \sum_{n=m}^{\alpha} v_n$$

$\therefore \sum_{n=m}^{\alpha} u_n >$ infinite (because ϵ ~~is~~ means tends to zero)

$\Rightarrow \sum_{n=m}^{\alpha} u_n$ is divergent

$\Rightarrow \sum_{n=1}^{\alpha} u_n$ is divergent.