CC-202 Differential Geometry ① Sarfaraz K.C.

Curves in space

Definition Differential Geometry In this branch of Mathematics we study the curves and surfaces with the help of differential calculus.

Space curve We can represent a space curve in the following two ways.

1. As intersection of two surfaces:

Let $f_1(x,y,z)=0$, $f_2(x,y,z)=0$ —— (1)

represent two surfaces then these two equations together represent the curve of intersection of these two surfaces. This curve will be called a plane curve if it lies on a plane, otherwise it is said to be a skew, twisted or tortuous.

2. Parametric representation:

If the coordinates of a point on a space curve be represented by the equations of the form $x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$ —— (2)

where f_1, f_2, f_3 are real valued funs of a single real variable t ranging over a set of values $a \le t \le b$. The Eq (2) represents as parametric Eq. of space curve.

3. Vector representation of a space curve:

If r be the p.v. of a current point P on the space curve whose cartesian coordinates are x, y, z then we know that

$$r = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

or $r = f(t)$ or $r = (f_1(t), f_2(t), f_3(t))$

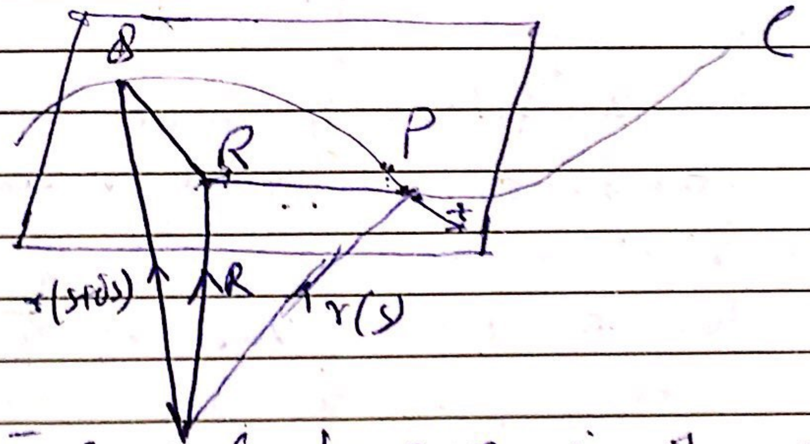
where f is a vector valued fun of a single variable t.

principal normal, Binormal and fundamental planes.

(a) principal normal: The principal normal at any point P of a given curve C is defined as the normal which lies in the osculating plane at P.

Osculating plane: Let Q and R be any two points on the curve which are close to P, then the limiting position of the plane PQR as the points Q and R tend to P is called the osculating plane at the point P.

From above we conclude that osculating plane has a three point contact or a contact of second order.



Let the curve C be given in terms of parameter s, by $r = r(s)$ where s is the length of the arc of the curve measured from a fixed point on it. The points P and Q correspond to parameters s and s+ds,

The Eq of the osculating plane in terms of parameter s of the point P.

$$[R - r(s), r'(s), r''(s)] = 0$$

unit vector along principal normal

⇒ ~~The unit vector along principal normal shall be denoted by n.~~

(b) Binormal: The binormal at any point P of a given curve C is defined as the normal which is ⊥r to the osculating plane.

unit vector along binormal: ~~The unit vector along the binormal shall be denoted by b.~~

(c) Directions of Principal normal and binormal:

~~The equation of the osculating plane is $(R-r)(\dot{r}\times\ddot{r})=0$. Since binormal is ⊥r to~~ osculating plane it is therefore parallel to $\dot{r}\times\ddot{r}$

Again principal normal is ⊥r to tangent $\dot{r}$ and also ⊥r to binormal $(\dot{r}\times\ddot{r})$ it is therefore parallel to $(\dot{r}\times\ddot{r})\times\dot{r}$.

ie $\dot{r}\cdot\dot{r}\,\ddot{r}-\dot{r}\cdot\ddot{r}\,\dot{r}$

ie $\dot{r}^2\ddot{r}-(\dot{r}\cdot\ddot{r})\dot{r}$

(d) Direction ratios of Principal normal and binormal

If $r=xi+yj+zk$ and dashes denote differentiation w.r.t. s and dots denote differentiation w.r.t. parameter then $r'=\Sigma x'i$, $r''=\Sigma x''i$,

$\dot{r}=\Sigma\dot{x}i$, $\ddot{r}=\Sigma\ddot{x}i$

Principal normal is parallel to r'' and hence its direction ratios are x'', y'', z''.

~~Further binormal is~~ parallel to $r'\times r''$

ie

$$(\Sigma x' i)\times(\Sigma x'' i) = \begin{vmatrix} i & j & k \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

Hence the direction ratios are

$$y'z''-y''z', \quad z'x''-x'z'', \quad x'y''-y'x''.$$

In terms of differentiation w.r.t. parameter t.

Binormal is parallel to $\dot{r}\times\ddot{r}$ and hence as above its direction ratios are

$$\dot y\ddot z-\ddot y\dot z, \quad \dot z\ddot x-\dot x\ddot z, \quad \dot x\ddot y-\dot y\ddot x$$

Principal normal is $\parallel$ to $(\dot r\times\ddot r)\times\dot r$

$$= \begin{vmatrix} i & j & k \\ \dot y\ddot z-\ddot y\dot z & \dot z\ddot x-\dot x\ddot z & \dot x\ddot y-\ddot x\dot y \\ \dot x & \dot y & \dot z \end{vmatrix}$$

Hence the drs are

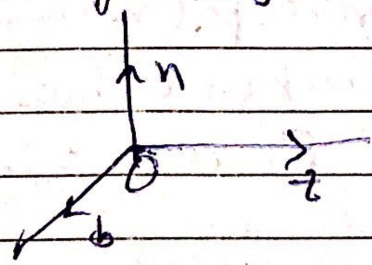
$$\dot z(\dot z\ddot x-\dot x\ddot z)-\dot y(\dot x\ddot y-\ddot x\dot y), \quad \dot x(\dot x\ddot y-\ddot x\dot y)-\dot z(\dot y\ddot z-\dot z\ddot y),$$
$$\dot y(\dot y\ddot z-\dot z\ddot y)-\dot x(\dot z\ddot x-\dot x\ddot z).$$

(5) The unit orthogonal vectors t, n, b.

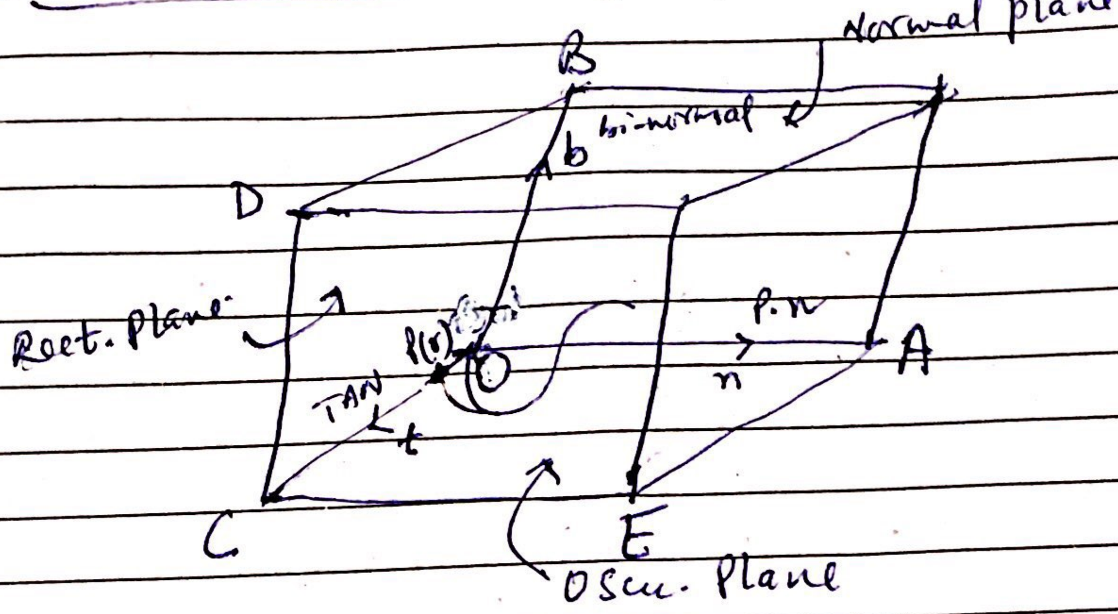
We know that principal normal and binormal are $\perp^r$ to each other and both these being normals are perpendicular to t. These three form a triad of mutually perpendicular unit vectors such that t, n, b form a right handed orthogonal system of axes.

$\therefore\ t\times n=b, \quad n\times b=t, \quad b\times t=n$

$t\cdot n=0, \quad n\cdot b=0, \quad b\cdot t=0$



(F) The three fundamental planes:



At each point of the curve there is a triad of orthogonal unit vectors which determine three fundamental planes as shown in the figure which contain two of these the third being the normal to that plane and which are mutually perpendicular.

Normal plane The plane through P containing b and n whose normal is t is called normal plane whose equation is $(R-r) \cdot t = 0$

Osculating plane The plane through P containing t and n whose normal is b is called osculating plane whose equation is $(R-r) \cdot b = 0$

Rectifying Plane: The plane through P containing b and t whose normal n is called rectifying plane. whose equation is

$$(R-r) \cdot n = 0$$

r is the p.v. of any point P on the curve and R that of any point on the respective plane