Article 5 :- State and prove Cauchy's root test

Statement :- If $\sum_{n=1}^{\infty} u_n$ be series of +ve terms then it is convergent if $\lim_{n\to\infty} (u_n)^{1/n} < 1$ divergent if $\lim_{n\to\infty} (u_n)^{1/n} > 1$ and test fail if $\lim_{n\to\infty} (u_n)^{1/n} = 1$

Proof :- Let $\lim_{n\to\infty} (u_n)^{1/n} = l$.

i.e for very small $\epsilon > 0$, $\exists$ a large +ve integer m such that

$$|(u_n)^{1/n} - l| < \epsilon \quad \forall\, n \geq m$$

$$\Rightarrow -\epsilon < (u_n)^{1/n} - l < \epsilon \quad \forall\, n \geq m$$

$$\Rightarrow l - \epsilon < (u_n)^{1/n} < \epsilon + l \quad \forall\, n \geq m$$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n \quad \forall\, n \geq m \quad \text{——— (1)}$$

Case I :- Let $l < 1$

From (1),

$$u_n < (l+\epsilon)^n \quad \forall\, n \geq m$$

$$u_m < (l+\epsilon)^m$$

$$u_{m+1} < (l+\epsilon)^{m+1}$$

$$u_{m+2} < (l+\epsilon)^{m+2}$$

Adding all

$$\sum_{n=m}^{\infty} u_n < (l+\epsilon)^m + (l+\epsilon)^{m+1} + (l+\epsilon)^{m+2} + \dots$$

= Sum of infinite G.P of c.r $= l+\epsilon < 1$ and So, it is convergent

$$\sum_{n \geq m}^{\infty} u_n < \text{finite}$$

$\Rightarrow \sum_{n \geq m}^{\infty} u_n$ is convergent

$\Rightarrow \sum_{n \geq 1}^{\infty} u_n$ is convergent

Case (II) :- $l > 1$

From (1)

$$u_n > (l-\epsilon)^n \quad \forall\, n \geq m$$

$\therefore\; u_m > (l-\epsilon)^m$

$u_{m+1} > (l-\epsilon)^{m+1}$

$u_{m+2} > (l-\epsilon)^{m+2}$

$\dots\dots$

$$\sum_{n=m}^{\infty} u_n > (l-\epsilon)^m + (l-\epsilon)^{m+1} + (l-\epsilon)^{m+2} + \dots$$

= Sum of infinite G.P of c.r $= l-\epsilon > 1$ and So, it is divergent

$\therefore \sum_{n\to\infty}^{\infty} u_n \to$ infinite

$\Rightarrow \sum_{n\to\infty}^{\infty} u_n$ is divergent

$\Rightarrow \sum_{n\to 1}^{\infty} u_n$ is divergent

Case (III) :- Let $l = 1$

Here test fails and which proved by following example :-

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ which is ~~ese~~ auxilliary series and it is convergent if $p > 1$ and divergent $p \leq 1$

i.e it depends on p

$$\because u_n = \frac{1}{n^p}$$

$$(u_n)^{1/n} = \frac{1}{n^{p/n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{p/n}} \quad \text{—— (2)}$$

Now, $\lim_{n \to \alpha} \frac{p}{n} \log n = \lim_{n \to \alpha} \frac{p \cdot \frac{1}{n}}{1}$ (By L.H rule)

$\Rightarrow \lim_{n \to \alpha} \frac{p}{n} \log n = \lim_{n \to \alpha} \frac{p}{n}$

$\Rightarrow \lim_{n \to \alpha} \frac{p}{n} \log n = 0$

$\Rightarrow \lim_{n \to \alpha} n^{p/n} = e^0 = 1$

$\therefore$ Equ^n (2) becomes,

$\lim_{n \to \alpha} (U_n)^{\frac{1}{n}} = \frac{1}{1} = 1$

$\therefore$ Hence the statement is proved

Q. If $\sum_{n=1}^{\alpha} U_n$, $U_n > 0 \ \forall n$ and $\lim_{n \to \alpha} (U_n)^{\frac{1}{n}} = l$ then prove that series is convergent if $l < 1$, divergent if $l > 1$ and test fail if $l = 1$

Ans:- $\lim_{n \to \alpha} (U_n)^{\frac{1}{n}} = l$

i.e, for very small $\epsilon > 0$ $\exists$ a large +ve integer m such that

$$|(U_n)^{\frac{1}{n}} - l| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow -\epsilon < (U_n)^{\frac{1}{n}} - l < \epsilon \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon < (U_n)^{\frac{1}{n}} < l + \epsilon \quad \forall n \geq m$$

$$\Rightarrow (l - \epsilon)^n < U_n < (l + \epsilon)^n \quad \forall n \geq m \qquad \text{--- (1)}$$

Case I :- Let $l < 1$

From (1),

$U_n < (l+\epsilon)^n \quad \forall n \geq m$

$\therefore U_m < (l+\epsilon)^m$

$U_{m+1} < (l+\epsilon)^{m+1}$

$U_{m+2} < (l+\epsilon)^{m+2}$

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Adding all,

$$\sum_{n=m}^{\alpha} U_n < (l+\epsilon)^m + (l+\epsilon)^{m+1} + (l+\epsilon)^{m+2} + \cdots$$

= sum of infinite G.P of c.r $= (l+\epsilon) < 1$ and so it is convergent

$$\sum_{n=m}^{\alpha} U_n < \text{finite}$$

$\therefore \sum_{n=m}^{\alpha} U_n$ is convergent.

$\therefore \sum_{n=1}^{\alpha} U_n$ is Convergent.

Case II :- Let $l > 1$

From ①,

$(l-\epsilon)^n < u_n \quad \forall\, n \geq m$

$\Rightarrow \quad u_n > (l-\epsilon)^n \quad \forall\, n \geq m$

$\therefore \quad u_m > (l-\epsilon)^m$

$u_{m+1} > (l-\epsilon)^{m+1}$

$u_{m+2} > (l-\epsilon)^{m+2}$

$- \; - \; - \; - \; -$

Adding all,

$$\sum_{n=m}^{\infty} u_n > (l-\epsilon)^m + (l-\epsilon)^{m+1} + (l-\epsilon)^{m+2} + \cdots$$

= sum of infinite G.P of c.r $= l-\epsilon$ and so it is divergent

$\therefore \sum_{n=m}^{\infty} u_n >$ infinite

$\Rightarrow \sum_{n=m}^{\infty} u_n$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent

Case III :- Let $l = 1$

Here test fails and which proved by following examples :-

Let $\sum_{n=1}^{\infty} (u_n) = \sum_{n=1}^{\infty} \frac{1}{n^p}$, which is auxiliary series and it is convergent if $p > 1$ and divergent if $p \leq 1$ i.e it depend on p

$\therefore \quad U_n = \frac{1}{n^P}$

$\therefore \quad (U_n)^{\frac{1}{n}} = \frac{1}{n^{P/n}}$

$\therefore \lim_{n \to \alpha} (U_n)^{\frac{1}{n}} = \lim_{n \to \alpha} \frac{1}{n^{P/n}}$ — ②

Now, $\lim_{n \to \alpha} \frac{P}{n} \log n = \lim_{n \to \alpha} \frac{P \cdot \frac{1}{n}}{1}$ (By L.H rule)

$\Rightarrow \lim_{n \to \alpha} \frac{P}{n} \log n = \lim_{n \to \alpha} \frac{P}{n}$

$\Rightarrow \lim_{n \to \alpha} \frac{P}{n} \log n = 0$

$\Rightarrow \lim_{n \to \alpha} n^{P/n} = e^0$

$\Rightarrow \lim_{n \to \alpha} n^{P/n} = 1$

$\therefore$ Eq ② becomes,

$\lim_{n \to \alpha} (U_n)^{\frac{1}{n}} = 1$

$\therefore$ Hence the statement proved

☀ De - Alembert (D' Alembert) ratio test

Q. State and Prove De - Alembert ratio test and when it is fail.

Ans:- **Statement** :- If $\sum_{n=1}^{\alpha} u_n$ be a series of +ve terms then series is convergent if $\lim_{n\to\alpha} \frac{u_n}{u_{n+1}} > 1$ and divergent if $\lim_{n\to\alpha} \frac{u_n}{u_{n+1}} < 1$.

Proof :- To Prove series is convergent if $\lim_{n\to\alpha} \frac{u_n}{u_{n+1}} > 1$ i.e, $\lim_{n\to\alpha} \frac{u_{n+1}}{u_n} < 1$ and divergent if $\lim_{n\to\alpha} \frac{u_n}{u_{n+1}} < 1$ i.e, $\lim_{n\to\alpha} \frac{u_{n+1}}{u_n} > 1$ and test fail when $\lim_{n\to\alpha} \frac{u_{n+1}}{u_n} = 1$

Let $\lim_{n\to\alpha} \frac{u_{n+1}}{u_n} = k$

i.e, for very small $\epsilon > 0$ $\exists$ a large +ve integer m such that

$$\left|\frac{u_{n+1}}{u_n} - k\right| < \epsilon \quad \forall\, n \geq m$$

$$\Rightarrow -\epsilon < \frac{u_{n+1}}{u_n} - k < \epsilon \quad \forall\, n \geq m$$

$$\Rightarrow k-\epsilon < \frac{u_{n+1}}{u_n} < k+\epsilon \quad \forall\, n \geq m \quad \text{—①}$$

Case 1 :- Let $k < 1$

from ①,

$$\frac{u_{n+1}}{u_n} < k+\epsilon \quad \forall\, n \geq m$$

$$\therefore \frac{u_{m+1}}{u_m} < k+\epsilon$$

$$\frac{u_{m+2}}{u_{m+1}} < k+\epsilon$$

$$\frac{u_{m+3}}{u_{m+2}} < k+\epsilon$$

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$$\therefore \sum_{n=m}^{\alpha} u_n = u_m + u_{m+1} + u_{m+2} + \cdots$$

$$= u_m\left[1+\frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_m} + \cdots\right]$$

$$= u_m\left[1+\frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m} + \cdots\right]$$

$$< u_m\left[1+(k+\epsilon) + (k+\epsilon)^2 + \cdots\right]$$

$= u_m$ [sum of infinite G.P of c.r = $k+\epsilon < 1$, and so it is convergent]

$$\therefore \sum_{n=m}^{\alpha} u_n < \text{finite}$$

$\Rightarrow \sum_{n=m}^{\alpha} u_n$ is convergent

$\Rightarrow \sum_{n=1}^{\alpha} u_n$ is convergent

Case II :- Let $k > 1$

from ①,

$$\frac{u_{n+1}}{u_n} > k - \epsilon \quad \forall\, n \geq m$$

$$\therefore \frac{u_{m+1}}{u_m} > k - \epsilon$$

$$\frac{u_{m+2}}{u_{m+1}} > k - \epsilon$$

$$\frac{u_{m+3}}{u_{m+2}} > k - \epsilon$$

$$- - - - - -$$

$$\therefore \sum_{n=m}^{\alpha} u_n = u_m + u_{m+1} + u_{m+2} + - - - -$$

$$= u_m\left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_m} + - - -\right]$$

$$= u_m\left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m} + - - -\right]$$

$$> u_m\left[1 + (k-\epsilon) + (k-\epsilon)^2 + - - -\right]$$

$= u_m$[sum of infinite G.P. of c.r $= k-\epsilon > 1$ and so it is divergent]

$$\therefore \sum_{n=m}^{\alpha} u_n > \text{infinite}$$

$\therefore \sum_{n=m}^{\alpha} u_n$ is divergent

$\therefore \sum_{n=1}^{\alpha} u_n$ is divergent

Case III :- When $k = 1$, then D'Alembert ratio test fail which proved by following examples :-

Let $\sum_{n=1}^{\alpha} u_n = \sum_{n=1}^{\alpha} \frac{1}{n^p}$, which is Auxiliary series and it is convergent if $p > 1$ and divergent if $p \leq 1$, it, it depend on p.

$$\therefore u_n = \frac{1}{n^p}$$

$$\therefore \cancel{(u_n)^{\frac{1}{n}} = \frac{1}{n^{p/n}}}$$

$$\therefore u_{n+1} = \frac{1}{(n+1)^p}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p}$$

$$= \left(\frac{n}{n+1}\right)^p$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^p$$

$$\therefore \lim_{n\to\alpha} \frac{u_{n+1}}{u_n} = \lim_{n\to\alpha}\left(\frac{1}{1+\frac{1}{n}}\right)^p$$

$$= \left(\frac{1}{1+0}\right)^p$$

$$= 1$$

Hence the statement is proved.

REDMI NOTE 9
AI QUAD CAMERA

Cauchy's condensation test

STATEMENT :- If $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ are two series of +ve term where a is integer, $a > 1$ and $f(n)$ decreases as n increases [or $f(1) > f(2) > f(3) \ldots\ldots$] than both series converge or diverge together.

Proof :-

$$
\begin{aligned}
& f(1) + f(2) + f(3) + \ldots\ldots\ldots\ldots f(a) \\
& + f(a+1) + f(a+2) + \ldots\ldots\ldots\ldots f(a^2) \\
& + f(a^2+1) + f(a^2+2) + \ldots\ldots\ldots\ldots f(a^3) \\
& + f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + f(a^{k-1}+1) + f(a^{k-1}+2) + \ldots\ldots\ldots f(a^k) \\
& + f(a^k+1) + f(a^k+2) + \ldots\ldots\ldots f(a^{k+1}) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty
\end{aligned}
$$

The no. of terms in first row $= a$
" " " " " 2nd row $= a^2 - a$
" " " " " 3rd row $= a^3 - a^2$
and so on

$\therefore$ The no. of term in $(k+1)^{th}$ row $= a^{k+1} - a^k$

Case I :- Since, $f(n)$ decreases as n increases
Hence, each term of $(k+1)^{th}$ row is greater than $f(a^{k+1})$

$$\therefore f(a^k+1) + f(a^k+2) + \ldots\ldots\ldots f(a^{k+1}) > (a^{k+1} - a^k) f(a^{k+1})$$

$f(a^k+1)+f(a^k+2)+\ldots\ldots f(a^{k+1}) > \left(1-\frac{1}{a}\right)a^{k+1} f(a^{k+1})$

Putting $k=0,1,2,\ldots\ldots$ successively we get,

$f(2)+f(3)+\ldots\ldots f(a) > \left(1-\frac{1}{a}\right)a\, f(a)$

$f(a+1)+f(a+2)+\ldots\ldots f(a^2) > \left(1-\frac{1}{a}\right)a^2 f(a^2)$

$f(a^2+1)+f(a^2+2)+\ldots\ldots f(a^3) > \left(1-\frac{1}{a}\right)a^3 f(a^3)$

....

Adding all

$\sum_{n=1}^{\infty} f(n) - f(1) > \left(1-\frac{1}{a}\right)\sum_{n=1}^{\infty} a^n f(a^n)$

If $\sum_{n=1}^{\infty} a^n f(a^n)$ is divergent than from above

$\sum_{n=1}^{\infty} f(n)$ is also divergent

Case II :- Again each term of $(k+1)^{th}$ row is less than $f(a^k)$.

$f(a^k+1)+f(a^k+2)+\ldots\ldots f(a^{k+1}) < (a^{k+1}-a^k) f(a^k)$

$f(a^k+1)+f(a^k+2)+\ldots\ldots f(a^{k+1}) < (a-1)\, a^k f(a^k)$

Put $k=0,1,2,3$ successively we get,

$f(2)+f(3)+\ldots\ldots+f(a) < (a-1) f(1)$

$f(a+1)+f(a+2)+\ldots\ldots+f(a^2) < (a-1)\, a\, f(a)$

$f(a^2+1)+f(a^2+2)+\ldots\ldots+f(a^3) < (a-1)\, a^2 f(a^2)$

....

Adding all

$\sum_{n=1}^{\infty} f(n) - f(1) < (a-1) f(1) + \cancel{\sum_{n=1}^{\infty} a^n f(a^n)}\,(a-1)\sum_{n=1}^{\infty} a^n f(a^n)$

$\sum_{n=1}^{\infty} f(n) < a f(1) + (a-1)\sum_{n=1}^{\infty} a^n f(a^n)$

If $\sum_{n=1}^{\infty} a^n f(a^n)$ is convergent than from above

$\sum_{n=1}^{\infty} f(n)$ is also convergent

$\therefore$ Hence the statement

REDMI NOTE 9
AI QUAD CAMERA

2nd auxilliary Series

Q. Prove that the series $\sum \frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $p\le 1$

Statement: – If $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ are two series of +ve term

Proof: – Let $\sum f(n) = \sum \frac{1}{n(\log n)^p}$

$$\therefore f(n) = \frac{1}{n(\log n)^p}$$

$$\therefore a^n f(a^n) = \cancel{a^n} \cdot \frac{1}{\cancel{a^n}(\log a^n)^p}$$

$$= \frac{1}{(\log a^n)^p}$$

$$= \frac{1}{(n\log a)^p}$$

$$= \frac{1}{(\log a)^p \cdot n^p}$$

$$\sum a^n f(a^n) = \frac{1}{(\log a)^p} \sum \frac{1}{n^p} \quad \text{—— (1)}$$

But the auxilliary series $\sum \frac{1}{n^p}$ is convergent if $p>1$ and divergent $p\le 1$

Hence from (1),

$\sum a^n f(a^n)$ is convergent if $p>1$ and divergent if $p\le 1$.

Hence, By cauchy's condensation test, the given series is convergent if $p>1$ and divergent if $p\le 1$

proved

Remember: –

(1) Let $\sum_{n=1}^{\infty} v_n$ be convergent than $\sum_{n=1}^{\infty} u_n$ is convergent if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

(2) Let $\sum_{n=1}^{\infty} v_n$ be divergent than $\sum_{n=1}^{\infty} u_n$ is divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Raabe's Test

Q. state and prove Raabe's Test :-

Statement :- If $\sum_{n=1}^{\infty} u_n$ be a series of +ve term than it is convergent if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n > 1$ and divergent if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n < 1$

Proof :- Let $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ which is auxilliary series and it is convergent if $p>1$ and divergent if $p\leq 1$

Case I :- Let $p>1$

$\therefore \sum_{n=1}^{\infty} v_n$ is convergent

$\therefore \sum_{n=1}^{\infty} u_n$ is convergent

if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

i.e if $\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$

i.e if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$

By Binomial factorisation

i.e if $\frac{u_n}{u_{n+1}} > 1 + p\cdot\frac{1}{n} + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n^2} + \frac{p(p-1)(p-2)}{\lfloor 3}\cdot\frac{1}{n^3} + \cdots$

i.e if $\frac{u_n}{u_{n+1}} - 1 > \frac{p}{n} + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n^2} + \frac{p(p-1)(p-2)}{\lfloor 3}\cdot\frac{1}{n^3} + \cdots$

i.e if $\left(\frac{u_n}{u_{n+1}} - 1\right)n > p + \frac{p(p-1)}{\lfloor 2}\cdot\frac{1}{n} + \frac{p(p-1)(p-2)}{\lfloor 3}\cdot\frac{1}{n^2} + \cdots$

i.e if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n > p$

i.e if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n > p > 1$

i.e if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n > 1$

Case II :- Let $p \leq 1$

$\therefore \sum_{n=1}^{\infty} v_n$ is divergent

$\therefore \sum_{n=1}^{\infty} u_n$ is divergent

if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Simillarly, As by case I

$\sum_{n=1}^{\infty} u_n$ is divergent

if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n < p \leq 1$

i.e if $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}-1\right)n < 1$

Hence the statement prove

Logarithmic test

Statement :- If $\sum_{n=1}^{\infty} u_n$ be series of +ve term than it is convergent if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} > 1$ and divergent if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} < 1$

Proof :- Let $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^p}$ which is auxilliary series and it is convergent if $p>1$ and divergent if $p\le 1$

Case I :- Let $p>1$

$\therefore \sum_{n=1}^{\infty} v_n$ is convergent

$\therefore \sum_{n=1}^{\infty} u_n$ is convergent

if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

i.e if $\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$

i.e if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$

i.e if $\log\left(\frac{u_n}{u_{n+1}}\right) > \log\left(1+\frac{1}{n}\right)^p$

i.e if $\log\left(\frac{u_n}{u_{n+1}}\right) > p\log\left(1+\frac{1}{n}\right)$

i.e if $\log\left(\frac{u_n}{u_{n+1}}\right) > p\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\right]$ [from formula $\log(1+x)$]

i.e if $\log\left(\frac{u_n}{u_{n+1}}\right) > p$

i.e if $\cancel{\lim} \, n\log\left(\frac{u_n}{u_{n+1}}\right) > p\left[1-\frac{1}{2n}+\frac{1}{3n^2}+\dots\right]$

i.e if $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) > p > 1$

i.e if $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) > 1$

Case II :- Let $p\le 1$

$\therefore \sum_{n=1}^{\infty} v_n$ is divergent

$\therefore \sum_{n=1}^{\infty} u_n$ is divergent

if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Similarly as by case I

$\sum_{n=1}^{\infty} u_n$ is divergent

i.e if $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) < p \le 1$

i.e if $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) < 1$

Hence the statement proved

De-Morgan and Bertrand Test

Statement:- If $\sum_{n=1}^{\infty} u_n$ be series of +ve term than it is convergent if $\lim_{n\to\infty}\left[\left(\frac{u_n}{u_{n+1}}-1\right)n-1\right]\log n > 1$ and divergent if $\lim_{n\to\infty}\left[\left(\frac{u_n}{u_{n+1}}-1\right)n-1\right]\log n \leq 1$

Proof:- Let $\sum v_n = \sum \frac{1}{n(\log n)^p}$ which is convergent if $p>1$ and divergent if $p\leq 1$

Case I:- Let $p>1$

$\therefore \sum_{n=1}^{\infty} v_n$ is convergent

$\therefore \sum_{n=1}^{\infty} u_n$ is convergent

if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

i.e if $\frac{u_n}{u_{n+1}} > \frac{1}{n(\log n)^p} \times (n+1)\{\log(n+1)\}^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left\{\frac{\log(n+1)}{\log n}\right\}^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left\{\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left\{\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$ [From form $\log mn = \log m + \log n$]

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \left(\frac{1}{n}-\frac{1}{2n^2}+\cdots\right)}{\log n}\right\}^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right\}^p$

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left\{1 + p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots\right\}$

i.e if $\frac{u_n}{u_{n+1}} > \Big[1 + p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

$+ \frac{1}{n} + \frac{p}{n}\left(\frac{1}{n\log n} + \frac{1}{2n^2\log n} + \cdots\right) + \cdots\Big]$

i.e if $\frac{u_n}{u_{n+1}} - \left(1+\frac{1}{n}\right) > p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

$+\frac{p}{n}\left(\frac{1}{n\log n} + \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

i.e if $\left(\frac{u_n}{u_{n+1}} - 1\right) - \frac{1}{n} > p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

$+\frac{p}{n}\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

i.e if $\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1 > p\left(\frac{1}{\log n} - \frac{1}{2n\log n} + \cdots\right) + \cdots$

$p\left(\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right) + \cdots$

i.e if $\left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right]\log n > p\left(1 - \frac{1}{2n} + \cdots\right) + \cdots$

$p\left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right) + \cdots$

i.e if $\lim_{n\to\infty} \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right] \log n > P > 1$

i.e if $\lim_{n\to\infty} \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right] \log n > 1$

Case-II :- Let $P \leq 1$

$\therefore \sum v_n$ is divergent

$\therefore \sum u_n$ is divergent

if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Similarly as by case (I)

$\therefore \sum_{n=1}^{\infty} u_n$ is divergent

if $\lim_{n\to\infty} \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right] \log n < P \leq 1$

i.e if $\lim_{n\to\infty} \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right] \log n < 1$

$\therefore$ Hence the statement proved

ARTICLE :-

Prove that the necessary condition that the series $\sum_{n=1}^{\infty} u_n$ is convergent if $\lim_{n\to\infty} u_n = 0$ (or $u_n \to 0$ as $u_n \to \infty$) but not sufficient

Let $\sum_{n=1}^{\infty} u_n$ be convergent

$$\therefore \sum_{n=1}^{\infty} u_n = \text{finite} = k \text{ (say)} \quad \text{—— (1)}$$

$$\therefore \lim_{n\to\infty} \sum_{n=1}^{n} u_n = \sum_{n=1}^{n-1} u_n = \sum_{n=1}^{\infty} u_n = k \quad \text{—— (2)}$$

$$\therefore u_n = \sum_{n=1}^{n} u_n - \sum_{n=1}^{n-1} u_n$$

Taking limit $n \to \infty$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{n=1}^{n} u_n - \lim_{n\to\infty} \sum_{n=1}^{n-1} u_n$$

$$\lim_{n\to\infty} u_n = k - k \quad [\text{By (2)}]$$

$$\lim_{n\to\infty} u_n = 0$$

Converse :- The sufficient part is not prove which proved by following example :-

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$

Which is auxilliary series and it is divergent as $P=1$

Here $u_n = \frac{1}{n}$

$$\lim_{n\to\infty} u_n = 0$$

GAUSS's Ratio test

Statement :- If $\sum_{n=1}^{\infty} u_n$ be series of +ve term and $\frac{u_n}{u_{n+1}} = a + \frac{b}{\lambda} + \frac{c}{x^2}$, $\lambda > 1$ then series is convergent if $a>1$, divergent if $a<1$ and if $a=1$ then series is convergent if $b>1$ and divergent if $b \leq 1$

Proof :— $\because \frac{u_n}{u_{n+1}} = a + \frac{b}{n} + \frac{c}{n^\lambda}$ —— (1)

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = a$$

Hence, By De-Alembert ratio test the series is convergent if $a>1$ and divergent if $a<1$ and test fail if $a=1$

When $a=1$

$$\frac{u_n}{u_{n+1}} = 1 + \frac{b}{n} + \frac{c}{n^\lambda}$$

$$\Rightarrow \left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{b}{n} + \frac{c}{n^\lambda}$$

$$\Rightarrow \left(\frac{u_n}{u_{n+1}} - 1\right)n = b + \frac{c}{n^{\lambda-1}} \quad \text{—— (2)}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}} - 1\right)n = b \qquad \left[\begin{matrix}\because \lambda>1 \\ \therefore \lambda-1>0\end{matrix}\right]$$

Hence, By Raabe's test
The series is convergent if $b>1$ and divergent if $b<1$ and fail if $b=1$

When $b=1$

$$\left(\frac{u_n}{u_{n+1}} - 1\right)n = 1 + \frac{c}{n^{\lambda-1}}$$

$$\Rightarrow \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right] = \frac{c}{n^{\lambda-1}}$$

$$\Rightarrow \left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right]\log n = c\,\frac{\log n}{n^{\lambda-1}}$$

$$\Rightarrow \lim_{n\to\infty}\left[\left(\frac{u_n}{u_{n+1}} - 1\right)n - 1\right]\log n = c \cdot \lim_{n\to\infty} \frac{\log n}{n^{\lambda-1}}$$

$$= c \cdot \lim_{n\to\infty} \frac{\frac{1}{n}}{(\lambda-1)n^{\lambda-2}} \quad \text{[By L-H. Rule}$$

$$= c \cdot \lim_{n\to\infty} \frac{1}{(\lambda-1)n^{\lambda-1}}$$

$= 0 < 1 \quad [\because \lambda > 1]$

By, De-Morgan and Bertrand test the series is divergent when $b = 1$

Hence, finally the given series is convergent if $a > 1$, divergent if $a < 1$ and if $a = 1$ then series is convergent if $b > 1$ divergent if $b \leq 1$

For Problems

1. Series → Alternating → Leibnitz test
2. Series → NOT Alternating

Cauchy's root test

$u_n = (\quad)^{-n^2}$

$(\quad)^{n^2}$

$(\quad)^{-n}$

$(\quad)^{n}$

Comparision test

$$u_n = \frac{n^a + n^{a-1} + 1}{n^b + 2n}$$

When max power of numerator and denominator are obtained

De-Alembert Ratio test

I. Problems on Alternating Series

Test the convergency of the series.

(i) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots\cdots$

(ii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots$

(iii) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots\cdots$

(iv) $\frac{1}{x+a} - \frac{1}{x+2a} + \frac{1}{x+3a} - \cdots\cdots$

(v) $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \cdots\cdots$

① Ans :- Let $|u_n| = n^{th}$ term $= \frac{1}{n\sqrt{n}}$

$$\therefore |u_{n-1}| = \frac{1}{(n-1)\sqrt{n-1}}$$

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$$