Linear Equation with Constant coefficient

Any Equation in the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X \quad \text{(1)}$$

is called a linear Equation of nth degree. where $P_1, P_2, \dots, P_n$ are constants. So it is called linear Equation with constant coefficient.

The above Equation has two types of solution.

Case (1) if the RHS of term (1) i.e $X = 0$. Then the above Equation has only a complete solution called complementry function (C.F.)

Case (2) if the RHS term of Eq (1) is not equal to zero i.e $X \neq 0$. Then we will get another solution called particular Integral (P.I.)

Then in this case the equation has a complete solution

i.e $y = C.F. + P.I.$

Working rule of finding the C.F.

Writing the Eq (1) as

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$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = 0 \quad (X = 0)$$

Where D stands for $\frac{d}{dx}$, D^2 stand for $\frac{d^2}{dx^2}$ etc.

For finding C.F. we equate only the Operator terms $= 0$.

$D^n + P_1 D^{n-1} + \cdots + P_n = 0$

For auxilliary Equation, replacing D by m.

i.e $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$

Factorising the above Equation in to the factors we will get the values of m.

Case (1) If $m = m_1, m_2, m_3, \cdots, m_n$

i.e all roots are different. Then the C.F. of Eq (1) is as

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$$

Case (2) If $m = m_1, m_1, m_2, m_3, \cdots$

i.e some roots are repeating. Then

The solution of the Eq (1) i.e C.F is, when two roots are repeating

$$y = (C_1 + C_2 x)e^{m_1 x} + C_3 e^{m_2 x} + \cdots$$

If Three roots are repeating then C.F is

$$y = (C_1 + C_2 x + C_3 x^2)e^{m_1 x} + C_4 e^{m_2 x} + \cdots$$

Case (3) If $m = \alpha \pm i\beta, m_1, m_2, m_3, \cdots$ etc

(Complex roots occur in pair always)

Then C.F. is

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{m_4 x} + \cdots$$

where $c_1, c_2, c_3, \ldots$ are arbitrary constants.

In this class we discuss only about the solutions of linear equation when RHS of Eq (1) $= 0$.
That means the solution of Eq (1) is only C.F.

Problem (1) Solve $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

As writing $\frac{d}{dx} \equiv D$ etc.

ie $\{D^2 + (a+b)D + ab\}y = 0$

$\Rightarrow$ For auxilliary equation, replacing D by m

ie $m^2 + (a+b)m + ab = 0$

$\Rightarrow m^2 + am + bm + ab = 0$

$\Rightarrow m(m+a) + b(m+a) = 0$

$\Rightarrow (m+a)(m+b) = 0$

Either $m = -a$, or $m = -b$

ie both the roots are different. Hence the complete solution ie C.F. of the above equation is

$y = C_1 e^{-ax} + C_2 e^{-bx}$, where C_1, C_2 are constants

Problem (2) Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Ans writing the given Equation as

$(D^2 + 6D + 9)y = 0$, The left side of the equation is zero. So we will get only C.F.

Auxiliary Equation is

$m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0$

Either $m = -3, -3$

C.F. is

$y = (C_1 + C_2 x)e^{-3x}$ Ans

Problem (3) Solve $\frac{d^4y}{dx^4} - 81y = 0$

Ans The given Equation is written as

$(D^4 - 81)y = 0$

Auxilliary Equation is

$m^4 - 81 = 0$

$\Rightarrow (m^2-9)(m^2+9) = 0$

$\Rightarrow (m+3)(m-3)(m^2 - 3^2 i^2) = 0$

$\Rightarrow (m+3)(m-3)(m+3i)(m-3i) = 0$

Either $m = -3, 3, -3i, 3i$

Hence the C.F. is

$$y = c_1 e^{-3x} + c_2 e^{3x} + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$$

$$y = c_1 e^{-3x} + c_2 e^{3x} + c_3 \cos 3x + c_4 \sin 3x,$$

where c_1, c_2, c_3, c_4 are arbitrary constants

Problem (4) $a^2 \frac{d^4 y}{dx^4} = \frac{d^2 y}{dx^2}$ (solve)

$$\Rightarrow (a^2 D^4 - D^2) y = 0$$

A.E. is $a^2 m^4 - m^2 = 0$

$$\Rightarrow m^2 (a^2 m^2 - 1) = 0$$

$$\Rightarrow m^2 (am + 1)(am - 1) = 0$$

Either $m = 0, 0, -\frac{1}{a}, \frac{1}{a}$.

C.F. is

$$y = (c_1 + c_2 x) e^{0x} + c_3 e^{-\frac{1}{a}x} + c_4 e^{\frac{1}{a}x}$$

$$y = c_1 + c_2 x + c_3 e^{-\frac{x}{a}} + c_4 e^{\frac{x}{a}},$$

where c_1, c_2, c_3, c_4 are constants.

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