Date | Page No.

Thm Prove that every convergent sequence is Cauchy sequence but converse is not true.

Proof Let $\{x_n\}$ be a convergent sequence in a metric space (X, d), and $\lim_{n \to \infty} x_n = x$.

$\Rightarrow$ For every $\epsilon > 0$ $\exists$ an integer $n_\epsilon > 0$ s.t.

$$d(x_n, x) < \epsilon/2 \quad \forall \; n \geq n_\epsilon \quad \text{---} (1)$$

Putting $n = m$ in (1) we have

$$d(x_m, x) < \epsilon/2 \quad \forall \; n \geq n_\epsilon \quad \text{---} (2)$$

Now
$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \quad \text{(Triangle inequality)}$$
$$= d(x_m, x) + d(x_n, x)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{(From (1) \& (2))}$$

$\Rightarrow$ $\{x_n\}$ is Cauchy. (Proved)

Now to show that converse is not true,
consider $X = R - \{0\}$
ie X is set of all non-zero real numbers
and let $d(x, y) = |x - y| \quad \forall \; x, y \in X$

Consider a sequence $\{x_n\}$ where $x_n = \frac{1}{n}$ $n \in Z^+$
where Z^+ is set of all positive integers
Let for $\epsilon > 0$ n_ϵ be an integer greater than $\frac{2}{\epsilon}$ s.t.
Now
$$d(x_m, x_n) = |x_m - x_n|$$
$$\leq |x_m| + |x_n|$$

Teacher's Signature..................

$< \frac{1}{m} + \frac{1}{n}$

$< \frac{\epsilon}{2} + \epsilon/2 = \epsilon \quad \forall\, m, n \geq n_\epsilon \quad (\because n_\epsilon > \frac{2}{\epsilon})$

Therefor $\{x_n\}$ is Cauchy sequence.

Also $\{x_n\} \to 0$ as $n \to \infty$ but $0 \in X$

Hence the sequence is not convergent in X.

So converse is not true.

Complete metric space

A metric space is said to be complete if every Cauchy sequence in it is convergent.

Ex ① The usual metric space (R, d) is complete.

② The metric space (Z, d) is complete where $d(m, n) = |m - n|$

Closed Sphere: Let x_0 be any point in metric space (X, d) and r is a real number greater than zero then a closed sphere of radius r, centre at x is given by

$$S_r[x_0] = \{x \in X \mid d(x, x_0) \leq r\}.$$

It is also called closed ball

Date			
Page No.			

Th^m Prove that in a metric space a closed sphere is a closed set.

Proof Let $S_r[x_0]$ is a closed sphere in a metric space X.
where $x_0 \in X$ and $r > 0$.

We want show that $S_r[x_0]$ is closed
ie equivalent to show that $S_r[x_0]^c$ is open.

Let $x \in S_r[x_0]^c$

$$\Rightarrow d(x, x_0) \not\leq r$$

$$\Rightarrow d(x, x_0) > r$$

$$\Rightarrow d(x, x_0) - r > 0$$

Let $r_1 = d(x, x_0) - r$ ——— (1)

Consider open sphere $S_{r_1}(x)$

Let $t \in S_{r_1}(x)$

$$\Rightarrow d(t, x) < r_1$$

Now $d(x, x_0) \leq d(x, t) + d(t, x_0)$

$$\Rightarrow d(x, x_0) < r_1 + d(t, x_0)$$

$$\Rightarrow \cancel{d(x, x_0)} < \cancel{d(x, x_0)} - r + d(t, x_0)$$

$$\Rightarrow d(t, x_0) > r$$

$$\Rightarrow t \notin S_r[x_0]$$

ie $t \in S_r[x_0]^c$

ie $S_{r_1}(x) \subseteq S_r[x_0]^c$

$\Rightarrow S_r[x_0]^c$ is open

so $S_r[x_0]$ is closed.

Teacher's Signature..................