Cantor's Intersection Theorem

Let (X,d) be a complete metric space and let $\{F_n\}$ a decreasing sequence of non-empty closed subsets so that $\delta(F_n) \to 0$ as $n \to \infty$ then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one element of X.

(Note: Decreasing sequence of non-empty subsets F_n means $F_1 \supseteq F_2 \supseteq F_3 \cdots \supseteq F_n$)

Pf^m We will first show that F can not contain more than one element of X

Let us suppose to the contradiction that F contains two distinct elements x & y of X

$$\text{ie } x, y \in F$$

$$\Rightarrow x, y \in \bigcap_{n=1}^{\infty} F_n$$

$$\Rightarrow x, y \in F_n \text{ for each } n = 1, 2, 3, \cdots$$

Also $d(x,y) \leq \sup\{d(\alpha,\beta) \mid \alpha, \beta \in F_n\} = d(F_n)$

Since $\delta(F_n) \to 0$ as $n \to \infty$

$$\Rightarrow d(x,y) = 0$$

$$\Rightarrow x = y \text{ (contradiction)}$$

Hence F can not contain more than one element of X

Now we will show that F is non-empty.

Since F_n is non-empty $\forall\ n = 1, 2, 3, \ldots$

We construct a sequence $\{x_n\}$ such that $x_n \in F_n\ \forall n$.

Since $\delta(F_n) \to 0$ as $n \to \infty$, for every $\epsilon > 0$ $\exists$ an integer $n_\epsilon > 0$ such that

$$\delta(F_n) < \epsilon \quad \forall\ n \geq n_\epsilon$$

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Let $m > n > n_\epsilon$

Since $F_m \subseteq F_n$

~~So~~ $x_m \in F_m \Rightarrow x_m \in F_n$ Also $x_n \in F_n$

Now

$$d(x_m, x_n) \leq \delta(F_n) < \epsilon \quad \forall\ m, n \geq n_\epsilon$$

$\Rightarrow \{x_n\}$ is Cauchy sequence.

Since X is complete metric space

$\Rightarrow \{x_n\}$ is convergent sequence

i.e. $\exists\ x \in X$ s.t. $\lim_{n \to \infty} x_n = x$.

We'll now show that $x \in F$.

If sequence $\{x_n\}$ has only finitely many distinct points, then x is that point which is repeated and $F_n \subseteq F_{n_0}$ $\forall\ n \geq n_0$

$\Rightarrow x \in F_{n_0}$

If sequence $\{x_n\}$ has infinitely many distinct points, then x is a limit point of set of points of sequence, hence limit point of F_{n_0} $\because \{x_n | n \geq n_0\} \subseteq F_{n_0}$

Since F_{n_0} is closed, $x \in F_{n_0}$

So x belongs to arbitrary closed set F_{n_0}

$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n$

(Proved)

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