

Associative law does not hold in  $R$   
Hence  $R$  is not a group.

⑤ Is the set of integers form a group under multiplication  
Ans: The inverse does not exist in the set of integers under multiplication.

Since,  $2 \in \mathbb{Z}$  then

if inverse  $\frac{1}{2} \notin \mathbb{Z}$

Hence the set of integers is not a group under multiplication.

WVV

⑥ In the following cases determine whether the given system described a group. In case not point out which of the group axioms fails to hold.

(i) The set of rational numbers other than 1 with operation such that  $a * b = a + b - ab$

(ii)  $G = \{a_i, 0 \leq i < 7 \text{ and } i \in \mathbb{N}\}$  for the operation  $\circ$  such that  $a_i \circ a_j = a_{i+j}$  if  $i+j < 7$  and  $a_i \circ a_j = a_{i+j-7}$  if  $i+j \geq 7$ .

① Let  $Q$  = Set of rational numbers

Let  $G = Q - \{1\}$

given,  $a * b = a + b - ab$  ;  $a, b \in G$

To prove that  $(G, *)$  is a commutative group

Let,  $a, b, c \in G$  are arbitrary

(i) closure property  $\rightarrow$

Let  $a, b \in G$  s.t.  $a \neq 1, b \neq 1$

~~$a * b \in G$  since~~

Taking  $a + b - ab = 1$

$$\Rightarrow a(1-b) - 1(1-b) = 0$$

$$\Rightarrow (a-1)(1-b) = 0$$

$$\Rightarrow a=1, b=1$$

Contradictory to the fact that  $a \neq 1, b \neq 1$

Hence  $a+b-ab \neq 1$  and therefore  $a+b-ab \in G$   
i.e.  $a * b \in G$

(ii) Associative law  $\rightarrow$

$$a * (b * c) = (a * b) * c \quad \forall a, b, c \in G$$

$$\text{for } (a * b) * c = (a + b - ab) * c$$

$$= (a + b - ab) + c - (a + b - ab)c$$

$$= a + b + c - ab - ac - bc + abc$$

$$\text{and } a * (b * c) = a * (b + c - bc)$$

$$= a + (b + c - bc) - a(b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc$$

$$\text{Hence } (a * b) * c = a * (b * c)$$

(iii) Commutative law  $\rightarrow$

$$\therefore a * b = \cancel{a+b-ab} \quad a + b - ab = b + a - ba \\ = b * a$$

~~Since,  $(G, *)$  is a commutative group.~~  
 $\therefore$  Commutative law hold in  $G$

(iv) Existence of identity element  $\rightarrow$

If  $e$  be the identity element in  $G$  then we have

$$a * e = a \quad \forall a \in G$$

$$\Rightarrow a + e - ae = a$$

$$\Rightarrow e - ae = 0$$

$$\Rightarrow e = 0$$

$$\Rightarrow e \in G$$

$$\text{for } +1-a \neq 0$$

Also prove that,

$$0 \times a = 0 + a = a + 0 = a$$

Also,  $0 \neq 1$  and hence  $0 \notin G$  then  $\nexists$  identity element  ~~$0 \in G$~~

(iv) Existence of inverse  $\rightarrow$

Let  $b$  be the inverse of  $a \in G$ , then  $b \times a = e$   ~~$ie b + a = 1$~~

$$\Rightarrow b = \frac{-a}{1-a} = \frac{a}{a-1} \quad \Rightarrow b + a - ba = 0$$

$$a \in G \Rightarrow a \neq 1 \Rightarrow a-1 \neq 0$$

$$\Rightarrow \frac{a}{a-1} \neq 1 \Rightarrow \frac{a}{a-1} \in G$$

Thus every element  $a \in G$  has its inverse.

~~$(G, \times)$~~  This proves that  $(G, \times)$  is an abelian group.

(ii) Evidently-

$$G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$$

$$a_i \circ a_j = \begin{cases} a_{i+j} & \text{if } i+j < 7 \\ a_{i+j-7} & \text{if } i+j \geq 7 \end{cases}$$

Thus composite table is as under:

$\circ$	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
$a_0$	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
$a_1$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_0$
$a_2$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_0$	$a_1$
$a_3$	$a_3$	$a_4$	$a_5$	$a_6$	$a_0$	$a_1$	$a_2$
$a_4$	$a_4$	$a_5$	$a_6$	$a_0$	$a_1$	$a_2$	$a_3$
$a_5$	$a_5$	$a_6$	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$
$a_6$	$a_6$	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$

Let  $a_i, a_j, a_k \in G$  be arbitrary

(i) Closure property - Since all the elements of this table are in  $G$ , so closure property hold in  $G$ , under the given operation.

(ii) Associative property  $\rightarrow a_i \circ (a_j \circ a_k) = (a_i \circ a_j) \circ a_k$  for  
if  $i+j+k=1$  then L.H.S = R.H.S both of this are  
~~equal to 0 or 1 according as  $1 \leq i \leq 7$  or  $i \geq 7$  if  $i=1$~~   
if  $1 \leq i \leq 7$  or  $i \geq 7$ .

(iii) Existence of identity - It is clear that  $a_0 \circ a_i = a_i$   
for each  $i=0,1,2,3,4,5,6$

This prove that  $a_0 \in G$  is the identity element for  $G$ .

(iv) Existence of Inverse  $\rightarrow$  If  $a_j$  is the inverse of  $a_i$  then  
we must have  $(a_i)^{-1} = a_j$  and  $a_i \circ a_j = a_0$  from table  
we see that inverse of  $a_0, a_1, a_2, a_3, a_4, a_5, a_6$  are resp.  
 $a_0, a_6, a_5, a_4, a_3, a_2, a_1$  and all inverse belongs to  $G$ .

Hence  $G$  is a group.

(7)  $\mathbb{Q}$  is set of rational number Prove that the set  
 $\{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$  forms a commutative group of infinite  
order with respect to addition.

Ans: Let

$$G = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$$

To prove that  $(G, +)$  is an abelian group of infinite order

Let  $x, y, z \in G$  be arbitrary then  $\exists a, b, c, d, p, q \in \mathbb{Q}$  such

$$\text{that } x = a+b\sqrt{2}, y = c+d\sqrt{2}, z = p+q\sqrt{2}$$

$$a, b, c, d \in \mathbb{Q} \Rightarrow -a, -b, -c, -d \in \mathbb{Q}$$



and  $a+c, b+d \in \mathbb{Q}$  for  $\mathbb{Q}$  is closed with respect to addition.

(i) Closure property -  $x, y \in G$   
 $\Rightarrow x+y \in G$

$$\begin{aligned}\text{for } x+y &= (a+b\sqrt{2}) + (c+d\sqrt{2}) \\ &= (a+c) + (b+d)\sqrt{2} \\ &= a' + b'\sqrt{2} \in G\end{aligned}$$

[where  $a' = a+c, b' = b+d \in \mathbb{Q}$ ]

$$\therefore x+y \in G$$

(ii) Associativity  $\rightarrow x+(y+z)$   ~~$x+(y+z)$~~

$$\begin{aligned}&= (a+b\sqrt{2}) + [(c+d) + (e+f)\sqrt{2}] \\ &= (a+c+e) + (b+d+f)\sqrt{2}\end{aligned}$$

[for  $+$  is associative in  $\mathbb{Q}$ ]

$$\begin{aligned}&= [(a+c) + (b+d)\sqrt{2}] + (e+f)\sqrt{2} \\ &= (x+y) + z\end{aligned}$$

(iii) Commutative law -

$$x+y = y+x$$

$$\begin{aligned}\text{for } x+y &= (a+b\sqrt{2}) + (c+d\sqrt{2}) \\ &= (a+c) + (b+d)\sqrt{2} \\ &= (c+a) + (d+b)\sqrt{2} \\ &= c+d\sqrt{2} + a+b\sqrt{2} \\ &= y+x\end{aligned}$$

(iv) Existence of identity!

$\therefore 0 = 0+0\sqrt{2} \in G$   
and which is identity element, b/c

$$\forall x \in G, x+0 = 0+x = x$$

Since  $+$  is commutative in  $\mathbb{Q}$ .

(v) Existence of inverse - Inverse of every element

$$x \in G \text{ is } -x \in G$$

$$\text{for } x = a+b\sqrt{2}$$

$$-x = -a-b\sqrt{2} \in G \quad \{\because -a, -b \in \mathbb{Q}\}$$