Baire's Category theorem:

Let (X,d) be a metric space and if $\{F_n\}$ is a sequence of closed subsets of X such that $X = \bigcup_{n=1}^{\infty} F_n$ then at least one F_n contains a closed sphere.

Proof Let us assume to the contradiction that no F_n contains a closed sphere.

We will first prove an auxiliary result.

Let F be a closed sphere which does not contain any closed sphere.

Let K be a closed sphere defined as

$$K = \{x \in X \mid d(x, x_0) \leq r\}$$

Consider $S = \{x \in X \mid d(x, x_0) \leq r/2\}$

$\Rightarrow \exists\, x_1 \in S$ such that $x_1 \notin F$.

Consider $W = \{x \in X \mid d(x, x_1) \leq r_1\}$, $r_1 < r/2$ s.t $W \cap F = \phi$

Let $x \in W \Rightarrow d(x, x_1) \leq r_1$

$\Rightarrow d(x, x_1) \leq r/2$ also $x_1 \in S \Rightarrow d(x_1, x_0) \leq r/2$

so $d(x, x_0) \leq d(x, x_1) + d(x_1, x_0)$

$\leq r/2 + r/2 = r$

$\Rightarrow x \in K$.

so $W \subseteq K$

So we get the result as follows

"For every closed set not containing any closed sphere and a closed sphere K, $\exists$ a closed sphere W disjoint from F which is contained in K"

Now the proof of main theorem.

Consider K is any arbitrary closed sphere of radius 2.

The by result we have derived; $\exists$ a closed sphere K_1 of radius $r_1 \leq 1$ s.t $K_1 \cap F_1 = \phi$ where $K_1 \subseteq K$

Now consider K_1 as closed spher $\exists$ a closed sphere K_2 of radius $r_2 \leq \frac{1}{2}$ s.t $K_2 \cap F_2 = \phi$ where $K_2 \subseteq K_1$

Proceeding this way we get a sequence of closed spheres $\{K_n\}$ or radius $r_n \leq \frac{1}{2^n}$ s.t $K_n \cap F_n = \phi$ for all $n = 1, 2, 3, \ldots$

where $K \supseteq K_1 \supseteq K_2 \ldots \supseteq K_n \supseteq \ldots$

Also $\delta(K_n) = \frac{1}{2^n} \to 0$ as $n \to \infty$

Hence by Cantor's Intersection Theorem $\bigcap_{n=1}^{\infty} K_n$ contains exactly one point x_0 of E

Thus $x_0 \in K_n$ for each $n = 1, 2, 3 \ldots$

$\Rightarrow x_0 \notin F_n$ for any $n = 1, 2, 3, \ldots$

$\Rightarrow x_0 \notin \bigcup F_n = X$

Which is a contradiction as $x_0 \in X$.

Hence the theorem.

Teacher's Signature……………………