and $-x+x = (-a+a) + (-b+b)\sqrt{2}$
$= 0$

$O(G) = \infty$ as G contains an infinite number of elem
have $(G, *)$ is a commutative group of infinite order

(8) Show that the set G of all real numbers or (rational numbers) other than -1 is a group w.r. to the operation defined $a*b = a+b+ab \ \forall a,b \in G$

Ans→ Let Q = set of real numbers.

Let, $G = Q - \{-1\}$
$= \{a, , a$ is the rational no. and $a \neq -1\}$

for arbitrary elements $a, b \in G$ we define
$$a*b = a+b+ab$$

To prove that $(G *)$ is a group

Let, $a, b, c \in G$ are arbitrary.

(1) closure property →
$$a, b \in G$$
$$\Rightarrow a*b \in G$$

for $a+b+ab = -1$
$$\Rightarrow (a+1) + b(1+a) = 0$$
$$\Rightarrow (a+1)(b+1) = 0$$
$$\Rightarrow a = -1, \ b = -1$$

Contrary to the facts that $a \neq -1$, $b \neq -1$

Hence $a+b+ab \neq -1$ and therefore
$$a+b+ab \in G$$
i.e $a*b \in G$

(ii) Associative law →

$$a*(b*c) = (a*b)*c \quad \forall\, a,b \in G$$

for,

$$a*(b*c) = a*(b+c+bc)$$
$$= (a+b+c) + a(b+c+bc) + bc$$

and,

$$(a*b)*c = (a+b+ab)*c$$
$$= (a+b+ab) + c + (a+b+ab)c$$
$$= (a+b+c) + ab + ac + bc + abc$$
$$a*(b*c) = (a*b)*c$$

(iii) commutative law → If it is asking for abelian group (but here not)

$$a*b = b*a$$
$$\Rightarrow a*b = a+b+ab = b+a+ba$$
$$= b*a$$

Since $(G,*)$ is commutative.

(v) Existence of identity element →

If e be the identity element in G then, we must have

$$a*e = a \quad \forall\, a \in G$$

This $\Rightarrow a+e+ae = a$

$\Rightarrow e+ae = 0$

$\Rightarrow e(1+a) = 0$

$\Rightarrow e = 0$ since $a \neq -1$

Observe that

$$0*a = a*0 = 0+a-a\cdot 0 = a$$

Also $0 \neq -1$ and hence of G this $\exists$ identity element $0 \in G$

(v) Existence of inverse. Let b be the inverse of $a \in G$, then
then $b * a = e$

ie $b + a + ab = e = 0$.

$\Rightarrow b(1+a) = -a$

$\Rightarrow b = \frac{-a}{1+a}$

$a \in G$ and $a+1$ also $\in G$ as $a+1 \neq 0$

$\Rightarrow \frac{-a}{a+1} \neq -1$

$\Rightarrow \frac{-a}{a+1} \in G$

Thus every element $a \in G$ has its inverse $\frac{-a}{a+1} \in G$ this prove that $(G, *)$ is a group (abelian also).

(1) Prove that the set of all even integers is an abelian group of ifinite order with respect to the addition composition.

Ans) Let $A = \{0, \pm 2, \pm 4, \pm 6, ---\}$ we verify the follwoin group axious.

(i) Closure property → Sum of two even integers is also an even integers so closure property hold in A.

(ii) Associative property → Since the even integers obey associative law under addition.

(iii) Existence of identity → The identity element is 0

Since $a + 0 = 0 + a$

$= a \ \forall a \in A$

(iv) Existence of inverse → The inverse of $a \in A$ is $-a \in A$
since, $a+(-a) = -a+a = 0$

(v) Commutative law → Since, even integers obey commutative law

Hence A is a abelian group under addition.

2) Is the set of even integers ~~including~~ excluding 0 is a group under addition.

Ans → Let $A = \{\pm 2, \pm 4, \pm 6, ----\}$ We verify group axioms

(i) Existence of identity element –
Since the identity element does not exist, hence the set of even integers including '0' is not a group.

3) Does the set of odd integers is a group under additio

Ans → Let $A = \{\pm 1, \pm 3, \pm 5\}$
Since, sum of two odd integers may be an even integer since $3+5=8$
So, closure property doesn't hold in A so it is not a group.

2014 3marks

Theorem: Prove that if H is a Subgroup of a group G, then $h \in H$ if and only if $Hh = H$ and $hH = H$.

Proof: Let e be the identity element.

If part: Let $h \in H$
To prove $Hh = H$ and $hH = H$.
Let a be any arbitrary element of H.
$\therefore a \in H, h \in H \Rightarrow ah \in H$ {∵ H is Subgroup}
∵ $a \in H \Rightarrow ah \in hH$
~~$\therefore h \in H \Rightarrow ah \in hH \Rightarrow ah \in H$~~
$\therefore hH \subseteq H$ ——— (1)

Again $a = (hh^{-1}a)$ {∵ $hh^{-1} = e$}
$= h(h^{-1}a) \in hH$ {∵ $h^{-1} \in H, a \in H \Rightarrow h^{-1}a \in H$}
$\therefore a \in H \Rightarrow a \in hH$
$\therefore H \subseteq hH$ ——— (2)

From (3) and (4)

$H = hH$

Similarly we can show $H = Hh$

Only if part: let $H = hH$

To prove $h \in H$

$\because e \in H \Rightarrow he \in hH \Rightarrow h \in hH \Rightarrow h \in H$ $\{\because H = hH\}$

Proved

MATRICS

Q Show that the set of all matrices $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a group with respect to matrice multiplication where a and b not be equal to zero.

Ans: Let M be the set of matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where at least a or b not equal to zero.

Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$

Where at least a or b not equal to zero and at least c or d not equal to zero.

$\therefore |A|$ = Determinant of A

$= a^2 + b^2 \neq 0$

$|B| = c^2 + d^2 \neq 0$

$\therefore |AB| = |A||B| \neq 0$

(i) Closure property → Let $A, B \in M$

$$AB = \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix}$$

$$= \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix}$$

$$= \begin{bmatrix} p & q \\ -q & +p \end{bmatrix}$$

Where

$|AB| \neq 0$ (shown above)

At least p or q is not equal to zero. $AB \in M$ closure property hold in M.

(ii) Associative property - Since, matrice product is associative so associative law hold in M.

(iii) Existence of identity element - The identity element of M is

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $AI = IA = A \quad \forall A \in M$

(iv) Existence of inverse - If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in M$

then $A^{-1} = \frac{\text{Adjoint } A}{|A|}$

Where $|A| \neq 0$

$\therefore \frac{1}{|A|} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in M$

$\therefore$ each element of M is invertable thus M is a group under multiplication.

NOTE→ To find adjoint of 2×2 matrix $\begin{bmatrix} a_1 & b \\ -b & a_2 \end{bmatrix}$

Also if $|A| \neq 0$ is called non singular.

Q> Let M be a set of 2×2 real matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $ad \neq 0$ Is G is abelian.

Ans→ Let M be the set of matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where $ad \neq 0$

Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$

Where $ad \neq 0$ and $pr \neq 0$

$\therefore |A| = ad \neq 0$

$|B| = pr \neq 0$

$\therefore |AB| = |A|\,|B| \neq 0$