Paper-C306, U.G., Sem-III
GROUP THEORY
By Dr. P.C. Banerjee
Dept. of Maths., KCC

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Group → If G be a nonempty set and $*$ be a binary operation defined in G then 'G' is called a group if the following axioms (Properties) are hold

(i) Closure properties-
ie $a, b \in G \Rightarrow a * b \in G$
$\forall a, b \in G$

(ii) Associative Properties-
ie $a, b, c \in G \Rightarrow (a*b)*c = a*(b*c)$
$\forall a, b, c \in G$

(iii) Existense of identity - If there exist an element $e \in G$ such that $a*e = e*a = a$ $\forall a \in G$, then identity element 'e' exists in G.

(iv) Existence of inverse - For each $a \in G$, there exists element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$ then, we say inverse exists in 'G'.

Note- If G be a group under binary operation $*$, then we write $(G, *)$ is a group.

Abelian Group (commutative) :- A group $(G, *)$ is said to be abelian if $a*b = b*a$ $\forall a, b \in G$

Semi group (Associative group) — A non empty set G is said be semigroup under binary operation $*$ if the following properties are hold.

(i) Closure properties ie $a*b \in G$ $\forall a, b \in G$

(ii) Associative properties. ie. $a*(b*c) = (a*b)*c$
$\forall a, b, c \in G$

① Art:- Prove that a Right and Left identy of a group are equal.

Proof:- Let $(G, *)$ be a group and e and e' be resp. Right and Left identity of G.

To prove $e = e'$

$\because$ e is a right identity

$\therefore$ $a * e = a \quad \forall\, a \in G$

$\Rightarrow e' * e = e' \quad \{\because e' \in G\}$ — ①

$\because$ e' is a left identity

$\therefore$ $e' * a = a \quad \forall\, a \in G$

$\Rightarrow e' * e = e \quad \{\because e \in G\}$ — ②

from ① and ②

$e = e'$ <u>Proved</u>

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② Art:- Prove that the right and left inverse of an eleme of a group are equal.

Proof:- Let $(G, *)$ be a group and $a \in G$. Let a^{-1}, a' be resp. the right and left inverse of a.

to prove $a^{-1} = a'$

Let e be the identity element of G

$\because$ a^{-1} right inverse of a

$a * a^{-1} = e$ — ①

$\because$ a' is left inverse of a

$\therefore$ $a' * a = e$ — ②

$a, a', a^{-1} \in G$ by associative property

$a' * (a * a^{-1}) = (a' * a) * a^{-1}$

$\Rightarrow a' * e = e * a^{-1} \quad$ {using ① and ②}

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$\Rightarrow a' = a^{-1}$ Proved

3. Art- Prove that the identity element of a group is unique.

Proof- Let $(G, *)$ be a group and e be it's identity element.

To prove e is unique —

If possible, let e' be another identity element of G. By the defination of identity.

$a * e = e * a = a \; \forall a \in G$ —— ①

and $a * e' = e' * a = a \; \forall a \in G$ —— ②

$\because e' \in G$ So from ①

$e' * e = e * e' = e'$ —— ③

$e * e' = e' * e = e$ —— ④

from ③ and ④

$e = e'$

Hence the identity element of a group is unique.

4- Art- Prove that in a group the inverse of an element is unique.

Proof- Let $(G, *)$ be a group and e be it's identity element.

Let $a \in G$ and a^{-1} be it's inverse.

To prove a^{-1} is unique

If possible, let a' be another inverse of a. By the defination of inverse.

$a * a^{-1} = a^{-1} * a = e$ —— ①

and $a * a' = a' * a = e$ —— ②

(Note- for inverse help associative law)

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$\because a, a', a^{-1} \in G$. So by Associative law.

$a^{-1} * (a * a') = (a^{-1} * a) * a'$

$\Rightarrow a^{-1} * e = e * a'$ {using ① and ②}

$\Rightarrow a^{-1} = a'$

Hence the inverse of a group is unique.

(5) Art- State and prove left and right concellation laws

or

If a, b, c be three element of a group then P.T

(i) $a * b = a * c \Rightarrow b = c$ [left cancellation]

(ii) $b * a = c * a \Rightarrow b = c$ [right cancellation]

Ans STATEMENT:-

If a, b, c are three elements of a group, the left and right concellation laws are respectivelly.

(i) $a * b = a * c \Rightarrow b = c$

(ii) $b * a = c * a \Rightarrow b = c$

Proof.

If $(G, *)$ be a group and e be its identity element.

Let a^{-1} be inverse of $a \in G$

$\therefore a * a^{-1} = a^{-1} * a = e$ —— (1)

(i) we have

$a * b = a * c$

$\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$

$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$ [By associative law]

$\Rightarrow e * b = e * c$ {using (1)}

$\Rightarrow b = c$

(ii) we have

$b * a = c * a$

$\Rightarrow (b * a) * a^{-1} = (c * a) * a^{-1}$

$\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1})$ [By A.i]

$\Rightarrow b * e = c * e$

$b = c$ Prove

Hence the statement.

Art ⑥ P.T. the inverse of the product of the elements of a group is product of their inverses in reverse order.

or,

If 'a' and 'b' be two elements of a group then prove that $(ab)^{-1} = b^{-1}a^{-1}$

Ans Let a and b be the two elements of the group G and e be the identity element of G.

To prove,

$$(ab)^{-1} = b^{-1}a^{-1}$$

Now,

$$(ab)(b^{-1}a^{-1}) = a\{b(b^{-1}a^{-1})\}$$
$$= a\{(bb^{-1})a^{-1}\} \quad \text{(by associative law)}$$
$$= a(ea^{-1}) \quad [\because bb^{-1} = b^{-1}b = e]$$
$$= aa^{-1}$$
$$= e \quad \text{— (1)} \quad [\because aa^{-1} = a^{-1}a = e]$$

Next

$$(b^{-1}a^{-1})(ab) = b^{-1}\{a^{-1}(ab)\}$$
$$= b^{-1}\{(a^{-1}a)b\}$$
$$= b^{-1}(eb)$$

$= e$ —— (2)

from (1) and (2)

$(ab)(b^{-1}a^{-1}) = (b^{-1}a^{-1})(ab) = e$

$\Rightarrow b^{-1}a^{-1}$ is the inverse of ab.

But $(ab)^{-1}$ is the inverse of ab and the inverse of an element is unique so

$(ab)^{-1} = b^{-1}a^{-1}$

(7) Art- Prove that the inverse of the inverse of an element of a group is the element itself.

or

If a be an element of a group then Prove that $(a^{-1})^{-1} = a$

Proof

Let $(G, *)$ be a group and e be the identity element.

Let $a \in G$ and a^{-1} be the inverse of a.

To prove $(a^{-1})^{-1} = a$

By the defination of inverse

$a * a^{-1} = a^{-1} * a = e$ —— (1)

$\Rightarrow (a * a^{-1})^{-1} = (a^{-1} * a)^{-1} = e^{-1}$

$\Rightarrow (a^{-1})^{-1} * (a^{-1}) = (a^{-1}) * (a^{-1})^{-1} = e$

$\Rightarrow a^{-1}$ is the inverse of $(a^{-1})^{-1}$

$\because$ inverse of an element is unique so

$(a^{-1})^{-1} = a$

(8) If a and b be the elements of a group 'G' then P.T. the eqn.

(i) $ax = b$ and