Continuous Mappings:

Let (X, d_1) & (Y, d_2) be two metric spaces and $f: X \to Y$ be a mapping of X into Y. f is said to be continuous at a point $c \in X$ if for every $\epsilon > 0$ $\exists \delta > 0$ such that

$$\text{every } x \in X \text{ s.t. } d_1(x, c) < \delta \Rightarrow d_2(f(x_2), f(c)) < \epsilon$$

The mapping f is said to be continuous on $A \subseteq X$ if it is continuous at every point of A.

Alternately: f is said to be continuous at $c \in X$ if for every open sphere $S_\epsilon(f(c))$ $\exists$ an open sphere $S_\delta(c)$ such that

$$x \in S_\delta(c) \Rightarrow f(x) \in S_\epsilon(f(c))$$
$$\text{or } f(S_\delta(c)) \subseteq S_\epsilon(f(c))$$

Q Let (X, d_1) & (Y, d_2) be two metric spaces and $f: X \to Y$ be a constant function, Prove that f is continuous.

Proof Let $c \in X$ be any arbitrary point
Let $\epsilon > 0$ be any real number $\exists$ $\delta > 0$ (say $\delta = 1$)

Since $f(x)$ is constant function

$$\Rightarrow f(x) = f(c)$$
$$\Rightarrow d_2(f(x), f(c)) = 0$$

Hence for any $x \in X$ such that $d_1(x, c) < \delta$

$$\Rightarrow d_2(f(x), f(c)) < \epsilon \quad (\because d_2(f(x), f(c)) = 0)$$

Hence f is continuous

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Equivalence of continuity & sequentially continuity

Thm Let (X, d_1) and (Y, d_2) be metric spaces and f be a mapping of X into Y. Then f is continuous at $c \in X$ iff for every sequence $\{x_n\}$ in X

$$\lim_{n\to\infty} x_n = c \Rightarrow \lim_{n\to\infty} f(x_n) = f(c)$$

ie f is continuous at c iff f is sequentially continuous at c.

Proof Let f be continuous at $c \in X$. Let $\{x_n\}$ be any sequence in X which converges to c.

We want to show that $\lim_{n\to\infty} f(x_n) = f(c)$

Since f is continuous at $x = c$
$\Rightarrow$ For every $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$d_1(x, c) < \delta \Rightarrow d_2(f(x), f(c)) < \epsilon \quad \text{--- (1)}$$

Also $\{x_n\}$ converges to c mean for every $\delta > 0$ $\exists$ an integer $n_0 > 0$ such that

$$n \geq n_0 \Rightarrow d_1(x_n, c) < \delta \quad \text{--- (2)}$$

Putting x_n in place of x in (1) we get

$$d_1(x_n, c) < \delta \Rightarrow d_2(f(x_n), f(c)) < \epsilon \quad \text{--- (3)}$$

~~From (2) & (3)~~

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From ② & ③

$$n \geq n_0 \Rightarrow d_2(f(x_n), f(c)) < \epsilon.$$

$$\Rightarrow \lim_{n \to \infty} f(x_n) = f(c)$$

This proves necessary part.

For sufficient part

Let $\lim_{n \to \infty} x_n = c \Rightarrow \lim_{n \to \infty} f(x_n) = f(c)$

We want to show that $f(x)$ is continuous at $x = c$. Let us assume to the contradiction that $f(x)$ is not continuous at $x = c$

$\Rightarrow \exists\ \epsilon > 0$ for any $\delta > 0$ & $x \in X$ such that

$$d_1(x, c) < \delta \text{ and } d_2(f(x), f(c)) \geq \epsilon$$

Let $x_n \in X$ be such that $d_1(x_n, c) < \frac{1}{n}$ and $d_2(f(x_n), f(c)) \geq \epsilon$, $n = 1, 2, 3, \ldots$

Here we have $x_n \to c$ as $n \to \infty$ but $f(x_n)$ does not approach to $f(c)$ which is a contradiction.

Hence $f(x)$ is continuous at $x = c$.