

i) Closure property - Let $A, B \in M$

$$AB = \begin{bmatrix} ar+0 & aq+bz \\ 0+0 & dz \end{bmatrix} = \begin{bmatrix} ar & aq+bz \\ 0 & dz \end{bmatrix} \\ = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

Where,

$$|AB| \neq 0 \text{ (shown above)}$$

$$\therefore xz \neq 0 \therefore AB \in M$$

\therefore closure property hold in M

ii) Associative property - Since matrix product is associative so associative law hold in M .

iii) Existence of identity - The identity element I is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Since } AI = IA = A \quad \forall A \in G$$

iv) Existence of inverse - If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

Now,

$$A^{-1} = \frac{\text{Adjoint } A}{|A|} \\ = \frac{1}{|A|} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix} \in M$$

Each element of M is invertible. Thus M is a group under multiplication. It remains to verify whether it is a abelian.

v) Commutative law \rightarrow Since matrices multiplication is not commutative. Hence M is not an abelian group under the matrix multiplication.

Q) Let G be a set of 2×2 real matrices $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ with $a \neq 0$ then (G, \cdot) is an abelian group. We verify the following axioms.

(i) closure property:-

for $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \in G$

We have, $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & (ba)^{-1} \end{bmatrix}$

$\because ab \neq 0$ as $a \neq 0, b \neq 0$ $= \begin{bmatrix} ab & 0 \\ 0 & (ab)^{-1} \end{bmatrix} \in G$ $\{ \because ab = ba \}$

Hence G is closed under matrix multiplication.

(ii) Associativity:- Matrices multiplication is associative.

(iii) Existence of identity:- The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in G is identity element since.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

(iv) Existence of inverse:- The matrix $\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ is the inverse of $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $= \begin{bmatrix} a^{-1} & 0 \\ 0 & (a^{-1})^{-1} \end{bmatrix} \in G$

Since, $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$
 $= \begin{pmatrix} aa^{-1} & 0 \\ 0 & a^{-1}a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a is real number thus 1 and a^{-1} are the identity element and inverse of $a \in \mathbb{R}$ under multiplication.

Therefore G is a group under multiplication.

It remains to show that G is abelian.

If a, b are two real numbers then

$$\left. \begin{aligned} ab &= ba \\ \Rightarrow a^{-1}b^{-1} &= b^{-1}a^{-1} \end{aligned} \right\} \text{--- (A)}$$

Now,

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} &= \begin{pmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} \\ &= \begin{pmatrix} ba & 0 \\ 0 & a^{-1}b^{-1} \end{pmatrix} \\ &= \begin{pmatrix} ba & 0 \\ 0 & b^{-1}a^{-1} \end{pmatrix} \quad \{\text{from (A)}\} \\ &= \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \end{aligned}$$

Hence G is abelian group under matrices multiplication.
Note - Matrix multiplication is not commutative.

Q> Show that the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ form a multiplicative abelian group.

Ans → We write,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G = \{A, B, C, D\}$$

We claim (G, \cdot) is an abelian group of order 4.

$$AA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \in G$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = B \in G$$

$$BD = DD = A, \quad CD = DA,$$

Similarly, $AC = C, AD = D, BA = A$ etc.

(i) Closure property → We can see that all elements in the composite table are the element of G and hence G is closed w.r. to matrices multiplication.

(ii) Associative law \rightarrow Multiplication is associative in G . Since associative law holds in case of matrix multiplication
i.e. $(AB)C = A(BC)$

(iii) Commutative law \rightarrow The entries in the first second, third and fourth columns of the composite table coincide with the corresponding entries in the first, second, third and fourth row,

it means that $PA = AP \forall P, A \in G$

(iv) Existence of identity \rightarrow From the composite table it is clear that $AA=A, AB=B, AC=C, AD=D$

This $\Rightarrow AP = P \forall P \in G$

$= AP = PA = P \forall P \in G$ [By (iii)]

"A is identity element in G

Thus \exists identity element in G . Also $A \in G$

(v) Existence of inverse $\rightarrow AA=A, BB=A, CC=A, DD=A$

This $\Rightarrow PP=A \forall P \in G$

$= P'P \Rightarrow P' \in G$

Hence $(G, *)$ is an abelian group.

Q) Prove that the set of matrices $A_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

Where α is real number from a group under multiplication.

Ans \rightarrow Let G denotes the set of matrices.

$$A_n = \begin{bmatrix} \cos n & -\sin n \\ \sin n & \cos n \end{bmatrix} \quad n \in \mathbb{R}$$

$$\text{i.e. } G = \{A_n : n \in \mathbb{R}\}$$

To prove that G is a group relative to matrices

multiplication.

$$A_n = \begin{pmatrix} \cos n & -\sin n \\ \sin n & \cos n \end{pmatrix}$$

$$A_m = \begin{pmatrix} \cos m & -\sin m \\ \sin m & \cos m \end{pmatrix} \quad n, m \in \mathbb{R}$$

(i) closure property - $A_n, A_m \in G$
 $\Rightarrow A_n A_m \in G$

for,

$$\begin{aligned} A_n A_m &= \begin{pmatrix} \cos n & -\sin n \\ \sin n & \cos n \end{pmatrix} \begin{pmatrix} \cos m & -\sin m \\ \sin m & \cos m \end{pmatrix} \\ &= \begin{pmatrix} \cos(n+m) & -\sin(n+m) \\ \sin(n+m) & \cos(n+m) \end{pmatrix} \\ &= A_{(n+m)} \in G \end{aligned}$$

for $n, m \in \mathbb{R} \Rightarrow n+m \in \mathbb{R}$

(ii) Associativity - $(A_n A_m) A_p = A_n (A_m A_p)$

Since, from matrix theory we know that the matrices multiplication satisfies the also associative law.

(iii) Existence of identity - Let I denotes unit matrices of second order then $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

also, I is expressed as

$$\begin{aligned} I &= \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} \\ &= A_0 \in G \quad \text{for } 0 \in \mathbb{R} \end{aligned}$$

From matrix theory we know that

$I A_n = A_n = A_n I \quad \forall A_n \in G$ Thus \exists identity element $I \in G$

Q. Existence of Inverse $\rightarrow \therefore A_n = \begin{pmatrix} \cos n & -\sin n \\ \sin n & \cos n \end{pmatrix}$

$$|A_n| = 1 \Rightarrow A_n \text{ is non singular}$$

$\Rightarrow A_n^{-1}$ exists

$$A_n^{-1} = \frac{\text{Adj } A_n}{|A_n|}$$

$$\Rightarrow A_n^{-1} = \begin{pmatrix} \cos n & \sin n \\ \sin n & \cos n \end{pmatrix} = \begin{pmatrix} \cos(-n) & -\sin(-n) \\ \sin(-n) & \cos(-n) \end{pmatrix}$$

$$= A(-n)$$

Thus $A_n^{-1} = A(-n) \in G$

for $n \in \mathbb{R} \Rightarrow -n \in \mathbb{R}$

Hence every matrix $A_n \in G$ has its inverse $A(-n) \in G$ this completes the proof.

Note-

$$(1) A_m, A_n = A_{m+n} = A_{n+m}$$

$$\text{for } m+n = n+m$$

$$= A_m A_n$$

This prove that G is an abelian group relative to matrix multiplication.

$$(2) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

Q. Show that the set of all $n \times n$ non singular (ie $|A| \neq 0$) matrices having their element are rational (real or complex) numbers is an infinite non abelian group