2/5/2020

HOMOGENEOUS LINEAR EQUATION

Any equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2}}{dx^{n-2}} y + \cdots + P_n y = 0 \text{ or } X$$

is called homogenous linear equation where $P_1, P_2, \ldots, P_n$ are constants and X is the fun of x

<u>W. R.</u> Put $\log x = z \Rightarrow x = e^z$

$$\frac{dz}{dx} = \frac{1}{x}, \text{ writing } \frac{d}{dz} = D.$$

where $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} = \frac{1}{x} \cdot \frac{dy}{dz}$

$$x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x \frac{d}{dx} \equiv \frac{d}{dz}$$

<u>Also</u> $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right)$

$$= \frac{1}{x} \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{1}{x^2} \cdot \frac{dy}{dz}$$

$$= \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} - \frac{1}{x^2} \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right].$$

$$x^2 \frac{d^2y}{dx^2} = (D^2 - D) y$$

$$x^2 \frac{d^2}{dx^2} \equiv D(D-1).$$

Similarly $x^3 \frac{d^3}{dx^3} = D(D-1)(D-2)$ and etc.

Problem (13) Solve the equation $x^2\frac{d^2y}{dx^2} + y = 3x^2$

Ans Given Eq. is $x^2\frac{d^2y}{dx^2} + y = 3x^2$

Put $\log x = z \Rightarrow x = e^z$

~~and writing $x\frac{d}{dx} \equiv D$ etc.~~

Then transformed equation is

$$D(D-1)y + y = 3e^{2z}$$

$$(D^2 - D + 1)y = 3e^{2z}$$

A.E. is $m^2 - m + 1 = 0$

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.1}}{2.1}$$

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

C.F. $y = e^{\frac{1}{2}z}\left(A\cos\frac{\sqrt{3}}{2}z + B\sin\frac{\sqrt{3}}{2}z\right)$

P.I. $y = \frac{3}{D^2 - D + 1}e^{2z} = \frac{3}{2^2 - 2 + 1}e^{2z}$

$$= \frac{3}{3}e^{2z} = e^{2z}$$

$\therefore$ Complete solution is

$$y = e^{z/2}\left(A\cos\frac{\sqrt{3}}{2}z + B\sin\frac{\sqrt{3}}{2}z\right) + e^{2z}$$

$$y = x^{\frac{1}{2}}\left[A\cos\left(\frac{\sqrt{3}}{2}\log x\right) + B\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x^2$$

Ans

Problem (14) Solve $(x^2D^3 - 4xD^2 + 6D)y = 4$

Ans Multiplying given equation by x,

$$\left(x^3\frac{d^3}{dx^3} - 4x^2\frac{d^2}{dx^2} + 6x\frac{d}{dx}\right)y = 4x$$

Put $\log x = z \Rightarrow x = e^z$, writing $x\frac{d}{dx} \equiv \frac{d}{dz} = D$ etc

$$\left[D(D-1)(D-2) - 4D(D-1) + 6D\right]y = 4e^z$$

$$\Rightarrow \left[D(D^2-3D+2) - 4D^2 + 4D + 6D\right]y = 4e^z$$

$$\Rightarrow \left[D^3 - 3D^2 + 2D - 4D^2 + 10D\right]y = 4e^z$$

$$\Rightarrow (D^3 - 7D^2 + 12D)y = 4e^z$$

A.E. $m^3 - 7m^2 + 12m = 0$

$$\Rightarrow m(m^2 - 7m + 12) = 0$$

$$\Rightarrow m(m-3)(m-4) = 0$$

Either $m = 0, 3, 4$

C.F. $y = c_1e^{0z} + c_2e^{3z} + c_3e^{4z}$

$$y = c_1 + c_2x^3 + c_3x^4$$

P.I. $y = \frac{1}{D^3 - 7D^2 + 12D}e^z = \frac{1}{1^3 - 7\cdot1^2 + 12\cdot1}e^z = \frac{e^z}{6}$

Complete solution is

$$y = c_1 + c_2x^3 + c_3x^4 + \frac{x}{6}$$ Ans

Problem (15) Solve $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4)$

Ans Put $x+1=u \Rightarrow \frac{du}{dx}=1$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{dy}{du}\cdot 1$$

$$\frac{d^2y}{dx^2}=\frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}=\frac{d}{du}\left(\frac{dy}{du}\right)\cdot 1=\frac{d^2y}{du^2}$$

The transformed equation is

$$u^2\frac{d^2y}{du^2}+u\frac{dy}{du}=\{2(u-1)+3\}\{2(u-1)+4\}$$

$$\Rightarrow \left\{u^2\frac{d^2}{du^2}+u\frac{d}{du}\right\}y=(2u+1)(2u+2)$$

It is homogenous Equation

Put $\log u=z \Rightarrow u=e^z$, writing $u\frac{d}{du}=D$ etc.

$$[D(D-1)+D]y=(2e^z+1)(2e^z+2)$$

$$\Rightarrow (D^2-D+D)y=4e^{2z}+4e^z+2e^z+2$$

$$\Rightarrow D^2y=4e^{2z}+6e^z+2$$

A.E. is $m^2=0 \Rightarrow m=0,0$

C.F $y=(C_1+C_2z)e^{0z}=(C_1+C_2\log u)$

$$=C_1+C_2\log(x+1)$$

P.I $y=\frac{4}{D^2}e^{2z}+\frac{6}{D^2}e^z+\frac{1}{D^2}(2)$

$$=e^{2z}+6e^z+2\cdot\frac{z^2}{2}$$

$$=u^2+6u+(\log u)^2=(x+1)^2+6(x+1)+\{\log(x+1)\}^2$$

$\therefore$ Complete solution is

$$y=C_1+C_2\log(x+1)+(x+1)^2+6(x+1)+\{\log(x+1)\}^2$$ Ans