Characterisation of continuous mappings in terms of open sets:

Th) Let (X, d_1) & (Y, d_2) be two metric spaces. A function $f: X \to Y$ is continuous if & only if $f^{-1}(U)$ is open subset of X for every open set U in Y.

Proof Necessary part:

Let f is continuous.

Let U be any open set in Y.

We have to prove that $f^{-1}(U)$ is open in X.

By defn
$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

If $f^{-1}(U) = \phi$ then it is open

So let $f^{-1}(U) \neq \phi$

Let $c \in f^{-1}(U) \Rightarrow f(c) \in U$

Since U is open $\Rightarrow$ We have some $\epsilon > 0$ such that
$$S_\epsilon(f(c)) \subseteq U$$

Also f is continuous at $x = c$ so $\exists\ \delta > 0$ s.t.
$$x \in S_\delta(c) \Rightarrow f(x) \in S_\epsilon(f(c))$$

ie $f(S_\delta(c)) \subseteq S_\epsilon(f(c)) \subseteq U$

ie $f(S_\delta(c)) \subseteq U$

$\Rightarrow S_\delta(c) \subseteq f^{-1}(U)$

$\Rightarrow f^{-1}(U)$ is open.

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Sufficient part: Let us suppose that inverse image of every open set is open. We want to prove that f is continuous.

Let $c \in X$ and consider $S_\epsilon(f(c))$ be any open sphere in Y. By our hypothesis $f^{-1}(S_\epsilon(f(c)))$ is an open set in X.

Also $c \in f^{-1}(S_\epsilon(f(c)))$

$\Rightarrow \exists\ \delta > 0$ such that $S_\delta(c) \subseteq f^{-1}(S_\epsilon(f(c)))$

$\Rightarrow f(S_\delta(c)) \subseteq f(f^{-1}(S_\epsilon(f(c)))) \subseteq S_\epsilon(f(c))$

Hence f is continuous.

(Proved)

Characterisation of continuous mappings in terms of closed sets:

Thm: Let (X, d_1) & (Y, d_2) be two metric spaces. A function $f: X \to Y$ is continuous if & only if $f^{-1}(G)$ is closed subset of X for every closed set G of X.

Proof: Necessary part: Let f be continuous function and let G be a closed set in Y

$\Rightarrow G^c$ is open

$\Rightarrow f^{-1}(G^c)$ is open in X ($\because f$ is continuous)

$\Rightarrow f^{-1}(G)^c$ is open in X ($\because f^{-1}(G^c) = f^{-1}(G)^c$)

$\Rightarrow f^{-1}(G)$ is closed in X

(Proved)

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Conversely (Sufficient part)

Let inverse image of f closed set closed. We want to prove the f is continuous.

Let U be any open subset of Y

$\Rightarrow$ U^c is closed subset of Y.

$\Rightarrow$ $f^{-1}(U^c)$ is closed subset of X (By our hypothesis)

$\Rightarrow$ $f^{-1}(U)^c$ is closed subset of X

$\Rightarrow$ $f^{-1}(U)$ is open subset of X

$\Rightarrow$ f is continuous.

(Proved)

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