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Some problems based on metric

Problem (1) Calculate first fundamental magnitudes for the surface $r = [u\cos v, u\sin v, f(u)]$.

Ans: The given surface is

$$r = [u\cos v, u\sin v, f(u)]$$

$$r_1 = \frac{\partial r}{\partial u} = [\cos v, \sin v, f'(u)]$$

$$r_2 = \frac{\partial r}{\partial v} = [-u\sin v, u\cos v, 0]$$

$$\therefore E = r_1^2 = \cos^2 v + \sin^2 v + f'(u)^2 = 1 + f'^2$$

$$F = r_1 \cdot r_2 = [\cos v, \sin v, f'(u)] \cdot [-u\sin v, u\cos v, 0]$$

$$= -u\cos v\sin v + u\cos v\sin v + 0 = 0$$

$$G = r_2^2 = (-u\sin v)^2 + (u\cos v)^2 + 0^2$$

$$= u^2(\sin^2 v + \cos^2 v) = u^2 \text{ Ans}$$

Problem (2) Show that the surface of revolution $x = u\cos v$, $y = u\sin v$, $z = f(u)$, the parametric curves form an orthogonal system and prove that $ds^2 = (1+f'^2)du^2 + u^2 dv^2$.

Ans: In problem (1), we find

$$E = 1 + f'^2, \quad F = 0, \quad G = u^2$$

Since $F = 0$, therefore the parametric curves are orthogonal.

[Deduction: If ω is the angle between the curves then $\tan\omega = \frac{H}{F}$, when $F = 0$, Then $\tan\omega = \infty$ and $\omega = 90^\circ$].

Again, $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$$= (1+f'^2)du^2 + 2\cdot 0\cdot du\,dv + u^2 dv^2$$

$$= (1+f'^2)du^2 + u^2 dv^2$$

Proved

Second Fundamental form

Let $r = r(u,v)$ be the equation of the surface and N be the unit normal vector to this surface at the point r. Then

$$N = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H} \qquad (\because H = |r_1 \times r_2|)$$

The quadratic differential form

$L\,du^2 + 2M\,du\,dv + N\,dv^2$ is called the second fundamental form of the surface in du, dv.

where $L = \frac{\partial^2 r}{\partial u^2} \cdot N = r_{11} \cdot N$

$$M = \frac{\partial^2 r}{\partial u \partial v} \cdot N = r_{12} \cdot N = r_{21} \cdot N$$

and $N = \frac{\partial^2 r}{\partial v^2} \cdot N = r_{22} \cdot N$.

The quantities L, M, N are called second fundamental coefficients or second order fundamental magnitudes.

Alternative forms for L, M, N

The vectors r_1 and r_2 are tangential to the surface at the point r, so the unit normal vector N is perpendicular to both the vectors r_1 and r_2, where $r_1 = \frac{\partial r}{\partial u}$ and $r_2 = \frac{\partial r}{\partial v}$.

Then we have $N \cdot r_1 = 0$, diffing w.r.t. u

$$N \cdot r_{11} + N_1 \cdot r_1 = 0 \Rightarrow N \cdot r_{11} = -N_1 \cdot r_1$$

$$\therefore L = N \cdot r_{11} = -N_1 \cdot r_1$$

Again diffing $N \cdot r_1 = 0$ w.r.t. v, we have

$$N \cdot r_{12} + N_2 \cdot r_1 = 0 \Rightarrow N \cdot r_{12} = -N_2 \cdot r_1$$

$$\therefore M = N \cdot r_{12} = -N_2 \cdot r_1$$

Also, we have $N \cdot r_2 = 0$, diffing w.r.t. u, we have

$$N \cdot r_{21} + N_1 \cdot r_2 = 0 \Rightarrow N \cdot r_{21} = -N_1 \cdot r_2$$

$$\therefore M = N \cdot r_{21} = -N_1 \cdot r_2$$

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Again diffing $N \cdot r_2 = 0$ w.r.t. v, we have

$$N_2 \cdot r_2 + N \cdot r_{22} = 0 \Rightarrow N \cdot r_{22} = -N_2 \cdot r_2$$

$$\therefore \; N = N \cdot r_{22} = -N_2 \cdot r_2$$

Thus we have

$$L = -N_1 \cdot r_1, \; M = -N_1 \cdot r_2 = -N_2 \cdot r_1 \text{ and } N = -N_2 \cdot r_2$$

Again $[r_1 \; r_2 \; r_{11}] = r_1 \cdot r_2 \times r_{11}$

$= r_1 \times r_2 \cdot r_{11}$

$= HN \cdot L \qquad \left(\because N = \dfrac{r_1 \times r_2}{H}\right)$

$= HL \qquad (\because N \text{ is unit vector})$

$[r_1 \; r_2 \; r_{12}] = r_1 \times r_2 \cdot r_{12}$

$= HN \cdot M = HM.$

and $[r_1 \; r_2 \; r_{22}] = r_1 \times r_2 \cdot r_{22} = HN \cdot N = HN$

* Equation of some important curves

1. Equation of surface of revolution is
$r = (u\cos\phi, u\sin\phi, g(u))$
2. Equation of sphere is
$r = (a\sin u\cos v, a\sin u\sin v, a\cos u)$
3. Equation of right circular cone is
$r = (u\cos v, u\sin v, u\cot\alpha)$, where α is semi vertical angle.
4. Equation of Anchor Ring is
$r = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u)$
5. Equation of General Helicoids is
$r = (f(u)\cos v, f(u)\sin v, g(u) + cv)$
6. Equation of Right Helicoid is
$r = (u\cos v, u\sin v, cv)$
7. Equation of the surface in Monge's form is
$r = (x, y, f(x, y))$

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