Thm Let (X, d_1) and (Y, d_2) be two metric spaces. Let $f: X \to Y$ be a mapping. Then following statements are equivalent:

(a) f is continuous at $x_0 \in X$

(b) Corresponding to each open set V containing $f(x_0)$ $\exists$ an open set U containing x_0 such that $f(U) \subseteq V$.

(c) For each open set V containing $f(x_0)$, $f^{-1}(V)$ is neighbourhood of x_0

(d) For each neighbourhood W of $f(x_0)$, $f^{-1}(W)$ is neighbour-hood of x_0

Proof (a) $\Rightarrow$ (b)

Let f is continuous at $x_0 \in X$

Let V is an open set containing $f(x_0)$

$\Rightarrow \exists\ \epsilon > 0$ such that $S_\epsilon(f(x_0)) \subset V$ —— ①

Since f is continuous, we have for every $\epsilon > 0$ $\exists\ \delta > 0$ such that

$$f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0)) \quad \text{—— ②}$$

Let $U = S_\delta(x_0)$. Clearly U is open set in X containing x_0.

$$f(U) \subseteq S_\epsilon f(x_0)$$

$$\Rightarrow f(U) \subseteq V \quad (\text{From ①})$$

(b) $\Rightarrow$ (a)

Let corresponding to each open set V containing $f(x_0)$ $\exists$ an open set U containing $f(x_0)$ such that $f(U) \subseteq V$

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We want to show that f is continuous at x_0

Let us consider $S_\epsilon(f(x_0)) = V$ —— (1)

Clearly V is open subset of Y containing x_0

By our hypothesis $\exists$ an open set U containing x_0 s.t.

$$f(U) \subset V \text{ —— (2)}$$

Since U is open set containing x_0, we have $\delta > 0$ s.t. $S_\delta(x_0) \subset U$

$$\Rightarrow f(S_\delta(x_0)) \subseteq f(U) \text{ —— (3)}$$

$$\Rightarrow f(S_\delta(x_0)) \subseteq V \quad \text{(From (2))}$$

$$\Rightarrow f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0)) \quad \text{(From (1))}$$

Hence f is continuous at x_0. (proved)

(b) $\Rightarrow$ (c)

Let corresponding to each open set V containing $f(x_0)$ $\exists$ an open set U containing x_0 such that $f(U) \subseteq V$.

We want to show that for each open set V containing $f(x_0)$, $f^{-1}(V)$ is nbd of x_0.

Let V is open set containing $f(x_0)$

$\Rightarrow$ $\exists$ an open set U containing x_0 s.t. $f(U) \subseteq V$

$$\Rightarrow U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(V)$$

Since U is open set containing x_0 and $U \subseteq f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is nbd of x_0 (proved)

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(c) $\Rightarrow$ (d) Let for every open set V containing $f(x_0)$, $f^{-1}(V)$ is nbd of x_0.

We want to show that for every nbd W of $f(x_0)$, $f^{-1}(W)$ is nbd of x_0

Let W is nbd of $f(x_0)$

$\Rightarrow \exists$ an open set V containing $f(x_0)$ s.t

$$V \subseteq W$$

$$\Rightarrow f^{-1}(V) \subseteq f^{-1}(W) \quad \text{——} (1)$$

By our hypothesis, as V is open set containing $f(x_0)$ $f^{-1}(V)$ is nbd of x_0

$\Rightarrow f^{-1}(W)$ is nbd of x_0 (By (1))

proved

(d) $\Rightarrow$ (b)

Let for every nbd W of $f(x_0)$, $f^{-1}(W)$ is nbd of x_0

Let V is an open set containing $f(x_0)$

$\Rightarrow V$ is nbd of $f(x_0)$

$\Rightarrow f^{-1}(V)$ is nbd of x_0

$\Rightarrow f^{-1}(V)$ is an open set containing x_0

(Hence proved)

Defn Uniform Continuity

Let (X, d_1) & (Y, d_2) be two metric spaces. A mapping $f: X \to Y$ is said to be uniformly continuous if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$d_1(x_1, x_2) < \delta \Rightarrow d_2(f(x_1), f(x_2)) < \epsilon.$$

Here choice of δ depends only on ϵ & not on any point of X.

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