5/2020

Linear Equation with variable Coefficient

The equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad (1)$

Where P, Q, R are functions of x, is called linear Equation with variable coefficient.

In this lecture we discuss about Normal form of the equation. When we remove the Ist derivative term of the equation. Then we call it normal form.

W.R. In Eq (1), we put $y = \rho u$, where ρ and u are both functions of x. Taking first and 2nd derivatives also. After simplifying, the Eq (1) reduces to the form

$$\frac{d^2y}{dx^2} + Q_1 u = R_1$$

when coefficient of Ist derivative term is equal to zero i.e. $\frac{2}{\rho}\frac{d\rho}{dx} + P = 0$

after integrating, $\rho = e^{-\frac{1}{2}\int P dx}$.

Where $Q_1 = Q - \frac{1}{2}\cdot\frac{dP}{dx} - \frac{1}{4}P^2$

and $R_1 = \frac{R}{\rho} = Re^{\frac{1}{2}\int P dx}$.

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Problem (16) Solve $\frac{d^2y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} + y = \frac{\sin 2x}{2}$.

Ans In the given equation

$$P = \frac{2}{x}, \quad Q = 1, \quad R = \frac{\sin 2x}{2}$$

$$\therefore \rho = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int \frac{2}{x}\,dx}$$

$$= e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

$$Q_1 = Q - \frac{1}{2} \cdot \frac{dP}{dx} - \frac{P^2}{4}$$

$$= 1 - \frac{1}{2} \cdot 2 \cdot \left(-\frac{1}{x^2}\right) - \frac{1}{4} \cdot \frac{4}{x^2}$$

$$= 1 + \frac{1}{x^2} - \frac{1}{x^2} = 1$$

$$R_1 = \frac{R}{\rho} = \frac{\frac{\sin 2x}{2}}{\frac{1}{x}} = \frac{1}{2} x \sin 2x$$

$\therefore$ The transformed equation is

$$\frac{d^2u}{dx^2} + Q_1 u = R_1$$

$$\Rightarrow \frac{d^2u}{dx^2} + 1 \cdot u = \frac{1}{2} x \sin 2x$$

$$\Rightarrow \frac{d^2u}{dx^2} + u = \frac{1}{2} x \sin 2x$$

$$\Rightarrow (D^2 + 1) u = \frac{1}{2} x \sin 2x$$

The A.E. is $m^2 + 1 = 0$

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$\Rightarrow m^2 = -1 \Rightarrow m^2 = i^2 \Rightarrow m = \pm i$

$\underline{C.F.}$ $u = e^{0x}(A\cos x + B\sin x)$

$\underline{P.I.}$

$$u = \frac{1}{D^2+1}\left(\frac{1}{2}x\sin 2x\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+1}\, I.P. \text{ of } e^{i2x}\cdot x$$

$$= \frac{1}{2} I.P. \text{ of } e^{i2x}\frac{1}{(D+2i)^2+1}(x)$$

$$= \frac{1}{2} I.P. \text{ of } e^{i2x}\frac{1}{D^2+4Di-4+1}(x)$$

$$= \frac{1}{2} I.P. \text{ of } e^{i2x}\frac{1}{D^2+4Di-3}(x)$$

$$= \frac{1}{2} I.P. \text{ of } e^{i2x}\frac{1}{-3\left\{1-\frac{D^2+4Di}{3}\right\}}(x)$$

$$= \frac{1}{2} I.P. \text{ of } e^{i2x}\cdot\left(-\frac{1}{3}\right)\left[1-\frac{D^2+4Di}{3}\right]^{-1}(x)$$

$$= -\frac{1}{6} I.P. \text{ of } e^{i2x}\cdot\left[1+\frac{D^2+4Di}{3}+\left(\frac{D^2+4Di}{3}\right)^2+\cdots\right](x)$$

$$= -\frac{1}{6} I.P. \text{ of } e^{i2x}\left[x+\frac{0+4\cdot1\cdot i}{3}+0\right]$$

$$= -\frac{1}{6} IP(\cos 2x + i\sin 2x)\cdot\left(x-\frac{4i}{3}\right)$$

$$u = -\frac{1}{6}\left(-\frac{4}{3}\cos 2x + x\sin 2x\right)$$
$$= \frac{2}{9}\cos 2x - \frac{1}{6}x\sin 2x$$

∴ The general solution is

$$u = A\cos x + B\sin x + \frac{2}{9}\cos 2x - \frac{1}{6}x\sin 2x$$

∵ $y = \rho u$

$$\Rightarrow y = \frac{1}{x}\left[A\cos x + B\sin x + \frac{2}{9}\cos 2x - \frac{1}{6}x\sin 2x\right]$$

$$\Rightarrow y = A\left(\frac{1}{x}\cos x\right) + B\left(\frac{1}{x}\sin x\right) + \frac{2}{9}\frac{\cos 2x}{x} - \frac{1}{6}\sin 2x,$$

where A and B are arbitrary constant.

Ans

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