Singular points of second order differential equation and their importance

In a second order linear differential equation

$$\frac{d^2y}{dx^2} + X_1(x)\frac{dy}{dx} + X_2(x)\,y = 0 :$$

If both X_1 and X_2 are analytic at $x = x_0$, then the point x_0 is called ordinary point of the differential equation. If either (or both) of these functions is not analytic at $x = x_0$, then x_0 is called the singular point of the differential equation.

Importance:

If the differential equation has no singularity then the equation can be solved by power series method

If the differential equation has an essential singularity, then it can not be solved by power series method.

If the differential equation has a removable singularity at $x = x_0$, so that $(x - x_0)X_1(x)$ and $(x - x_0)X_2(x)$ remains finite at $x = x_0$, then the equation can be solved by power series method.

Forbenius' method:

Let us consider a second order homogeneous differential equation:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$

We suppose that this equation has no solution

which is expressible as a finite linear combination of known elementary functions.
we may assume that it has a solution which can be expressed in the form of an infinite series like,

$$y = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad \text{——— (1)}$$

where $c_0, c_1, c_2, \ldots$ are constants, and there is a regular singular point at $x = x_0$.

An expression of this type is called a power series in $(x-x_0)$.

$$\therefore \frac{dy}{dx} = c_1 + 2c_2(x-x_0) + 3c_3(x-x_0)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n c_n (x-x_0)^{n-1} \quad \text{——— (2)}$$

similarly, $$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n (x-x_0)^{n-2} \quad \text{——— (3)}$$

Putting these expressions in the differential equation we get:

$$a_0 \sum_{n=2}^{\infty} n(n-1) c_n (x-x_0)^{n-2} + a_1 \sum_{n=2}^{\infty} n c_n (x-x_0)^{n-1} + a_2 \sum_{n=0}^{\infty} c_n (x-x_0)^n = 0$$

$$\Rightarrow K_0 + K_1(x-x_0) + K_2(x-x_0)^2 + \cdots = 0 \quad \text{——— (4)}$$

where K_0, K_1, K_2 are the functions of certain co-efficients c_n.

The equation (4) is valid for all x if the co-efficient of every power of x is zero.

$$\therefore K_0 = K_1 = K_2 = \cdots = 0$$

From these equations, the coefficients $C_0, C_1, C_2, C_3, \ldots$ can be calculated and hence the required solution of the given equation may be obtained.