Date | Page No.

Extension Theorem on uniform Continuity.

Thm Let (X, d_1) be a metric space and (Y, d_2) be a complete metric space. Let A be a dense subspace of X. If f is uniformly continuous mapping of A into Y, then f can be extended uniquely to a uniformly continuous mapping g of X into Y.

Proof Since A is dense in X. $\Rightarrow \bar{A} = X$

Hence every point of X is limit of sequence of elements of A. We define g as follows.

If $x \in A$ then $g(x) = f(x)$

If $x \notin A$ then take a sequence $\{x_n\}$ such that $x_n \in A$ and $\lim_{n\to\infty} x_n = x$.

~~Since f is uniformly continuous mapping of A into Y~~

$\Rightarrow$ for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon \quad \text{--- (1)}$$

Since $\{x_n\}$ is convergent sequence, it is Cauchy

$\Rightarrow$ $\exists$ an integer $n_0 > 0$ s.t

$$d_1(x_m, x_n) < \delta \quad \forall\ m, n \geq n_0$$

$$\Rightarrow d_2(f(x_m), f(x_n)) < \epsilon \quad \forall\ m, n \geq n_0$$

(From (1))

$\Rightarrow \{f(x_m)\}$ is a Cauchy sequence in Y.

Since Y is complete $\Rightarrow \lim_{n\to\infty} f(x_n)$ exists.

Let $g(x) = \lim_{n\to\infty} f(x_n)$

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We will show that for any other sequence $\{z_n\}$ of A converging to x, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n)$

Since $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} z_n = x$

$\Rightarrow d(x_n, z_n) \to 0$ as $n\to\infty$.

$\Rightarrow$ for any $\delta > 0$ $\exists$ an integer n_1 such that

$d_1(x_n, z_n) < \delta \quad \forall \; n \geq n_1$

$\Rightarrow d_2(f(x_n), f(z_n)) < \epsilon \quad \forall \quad n \geq n$.

Hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n)$.

~~Now~~ Hence $g(x)$ is unique for every $x \notin A$.

We will now show that g is uniformly continuous.

$\Rightarrow$ ~~Let~~ For $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$d(x_1, x_2) < \delta \Rightarrow d_2(f(x_1), f(x_2)) < \epsilon$. —— (1)

Let x' and x'' be two points of X such that

$d_1(x', x'') < \delta/2$ —— (2)

It is sufficient to show that $d_2(f(x'), f(x'')) < \epsilon$

Let $\{x'_n\}$ & $\{x''_n\}$ be two sequences in A converging to x' and x'' respectively.

So let for $\delta > 0$ $\exists$ $n_2 > 0$ s.t

$d_1(x'_n, x') < \delta/4 \quad \forall \; n \geq n_2$ —— (3)

Similarly for $\delta > 0$ $\exists$ $n_3 > 0$ s.t

$d(x''_n, x'') < \delta/4 \quad \forall \; n \geq n_3$ —— (4)

Let $n_4 = \max\{n_2, n_3\}$ then for $n \geq n_4$
(i.e $n \geq n_2$ & $n \geq n_3$) We have

$$d_1(x_n', x_n'') \leq d_1(x_n', x') + d_1(x', x'') + d_1(x'', x_n'')$$
$$< \delta/4 + \delta/2 + \delta/2 = \delta.$$
(From ② ③ & ④)

$\Rightarrow d_2(f(x_n'), f(x_n'')) < \epsilon \quad \forall n \geq n_4$ (From ①)

Also by defn of g; $f(x_n') \to g(x')$ & $f(x_n'') \to g(x'')$

Hence $d_2(f(x_n'), f(x_n'')) \longrightarrow d_2(g(x'), g(x''))$

$$\Rightarrow d_2(g(x'), g(x'')) = \lim_{n \to \infty} d_2(f(x_n'), f(x_n'')) < \epsilon.$$

Thus g is uniformly continuous.
Now we want to prove that g is unique.
Let if possible g_1 & g_2 be two different uniformly continuous extensions of f.

$$\Rightarrow g_1(x) = g_2(x) = f(x) \quad \forall x \in A$$

& If $x \notin A$, We can have sequence $\{x_n\}$ in A
Such that $\lim_{n \to \infty} x_n = x$

Since Uniformly continuous mapping is continuous
$\therefore g_1(x) = \lim_{n \to \infty} g_1(x_n)$ & $g_2(x) = \lim_{n \to \infty} g_2(x_n)$

By defn $g_1(x_n) = g_2(x_n) = f(x_n)$

$\Rightarrow g_1(x) = g_2(x) = \lim_{n \to \infty} f(x_n)$ (Contradiction)

Hence g is unique.

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