5/5/2020 8AM

Linear Differential Equation with variable coefficients

Let $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ —(1) is a differential Eq.

where P, Q, R are functions of x.

Method of solving the above Equation by changing the independent variable.

Let $z = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$

and $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dz}\cdot\dfrac{d^2z}{dx^2} + \dfrac{d^2y}{dz^2}\cdot\left(\dfrac{dz}{dx}\right)^2$

Putting these values in Eq (1). Then transformed equation is $\dfrac{d^2y}{dz^2} + P_1\dfrac{dy}{dz} + Q_1 y = R_1$ —(2), where

$$P_1 = \frac{P\dfrac{dz}{dx} + \dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2},\ Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

Let us choose z, s.t. $P_1 = 0$, then

$\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx} = 0$, Now put $\dfrac{dz}{dx} = v$.

From this we will get $z = \int e^{\int -P\,dx}\, dx$ —(3)

Then Eq (2) reduces in the form

$\dfrac{d^2y}{dz^2} + Q_1 y = R_1$. If Q_1 is constant.

Then the above diff. Equation reduces to linear diff. Eq. with constant coefficient. Again if Q_1 comes out of the form k/z^2. Then the above equation reduces to the form of a homogenous Equation.

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Problem No: 17 Solve the equation

$$\sin^2 x \frac{d^2y}{dx^2} + \sin x \cos x \frac{dy}{dx} + y = 0$$

Ans The given Equation is

$$\sin^2 x \frac{d^2y}{dx^2} + \sin x \cos x \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{\sin x \cos x}{\sin^2 x} \frac{dy}{dx} + \frac{1}{\sin^2 x} \cdot y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + \operatorname{cosec}^2 x \cdot y = 0$$

Here $P = \cot x$, $Q = \operatorname{cosec}^2 x$, $R = 0$

Changing the independent variable from x to z, the equation reduces to the form

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = 0 \quad \text{--- (1)}$$

Where $P_1 = \dfrac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$, $R_1 = 0$

Let us choose z, such that $P_1 = 0$,

then $P\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0$

$$\Rightarrow \cot x \frac{dz}{dx} + \frac{d^2z}{dx^2} = 0$$

Putting $\frac{dz}{dx} = v$, then $\cot x \cdot v + \frac{dv}{dx} = 0$

$$\Rightarrow \frac{dv}{dx} = -v \cot x \Rightarrow \frac{dv}{v} = -\cot x \, dx$$

$$\Rightarrow \int \frac{dv}{v} = -\int \cot x \, dx \Rightarrow \log v = -\log \sin x$$

$$\Rightarrow \log v = \log(\sin x)^{-1}$$

$$\Rightarrow v = (\sin x)^{-1} = \operatorname{cosec} x$$

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Now $\frac{dz}{dx} = q = \operatorname{cosec} x \Rightarrow \int dz = \int \operatorname{cosec} x \, dx$

$$\Rightarrow z = \log \tan \frac{x}{2} \quad \text{--- (2)}$$

Now $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \dfrac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 1$

Hence the equation (1) reduces to

$$\frac{d^2y}{dz^2} + 0.\frac{dy}{dz} + 1.y = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = 0$$

$$\Rightarrow (D^2+1)y = 0 \quad , \text{where } D \equiv \frac{d}{dz}$$

A.E. $\quad m^2 + 1 = 0$

$$\Rightarrow m^2 = -1 = i^2$$

$$\Rightarrow m = \pm i$$

C.F $\quad y = e^{0z}(A\cos z + B\sin z)$

$$= A\cos\left(\log\tan\frac{x}{2}\right) + B\sin\left(\log\tan\frac{x}{2}\right) \quad \text{by (2)}$$

Ans