(1) Do the set of all reside classes module 7 from a group w.r.to addition.

Ans → Let Z_7 denotes the set of all reside classes module 7, so that

$$Z_7 = \{[0], [1], [2], [3], ---- [6]\}$$

To determine the nature of the system $(Z_7, +)$

Putting $m=7$ in the 1 we get the solⁿ of $O(G)=7$

Since Z_7 contains elements finally we have proved that $(Z_7, +)$ is an abelian group of order 7.

MAPING

HOMOMORPHISM , ISOMORPHISM AND AUTOMORPHISM

HOMOMORPHISM → If (G, o) and $(G', *)$ be two groups, then the mapping of $f: G \to G'$ is called homomorphism.

$$\text{If } f(a \circ b) = f(a) * f(b) \cdot \forall a,b \in G$$

EPIMORPHISM → If (G, o) and $(G', *)$ be two groups, then the mapping $f: G \to G'$ is called epimorphism. If f is **onto** and $f(a \circ b) = f(a) * f(b)$

MONOMORPHISM → If (G, o) and $(G', *)$ be two groups then the one-one mapping $f: G \to G'$ is called monomorphism if

$$f(a \circ b) = f(a) * f(b) \quad \forall a,b \in G$$

ENDOMORPHISM - The homomorphism $f: G \to G'$ is called endomorphism. Ex- Let $(I, +)$ be the group of integers and $(G, \times)$ be a multiplication group of all integrals proes of I then a mapping $f: I \to G$ defined by $f(m) = 2^m$ is homomorphism

Since $f(m+n) = 2^{m+n}$
$= 2^m 2^n = f(m) \times f(n) \quad m, n \in I$

Isomorphism → If (G, o) and $(G', *)$ be two groups then one-one onto mapping $f: G \to G'$ is called isomorphism if $f(aob) = f(a) * f(b) \ \forall a, b \in G$

Ex →

Let $(R, \times)$ be the set of real numbers and is a group under multiplication and $(R, +)$ is a group under addition of real numbers. Thus a mapping $f: R \to R$ defined by $f(x) = \log x$ is Isomorphism.

NOTE → $f(x \times y) = \log(xy) = \log x + \log y = f(x) + f(y)$

(i) If $f: G \to G'$ is Isomorphism then G' is called Isomorphic Image of G and also write

$$G \approx G' \quad [\simeq \text{ or } \approx \text{ wiggler } \cong]$$

or $G \cong G'$

(ii) If $f: G \to G'$ is homomorphism, then $f(G) \subseteq G'$ is called homomorphic image of G

~~sign of homomorphism~~

AUTOMORPHISM → The Isomorphism $f: G \to G$ from the group G to itself is called automorphism.

Ex- G be a group, then the identity mapping
$I(x) = x \ \forall x \in G$ is automorphism.

INNER AUTOMORPHISM → If G be a group and for each element $a \in G$, Then the mapping $f_a : G \to G$ is called inner automorphism if $f_a(x)$ ~~[illegible]~~ $= a^{-1} x a \quad \forall \, x \in G$

OUTER AUTOMORPHISM → ~~The automorphism is~~ which is not inner automorphism is called outer automorphism.

VVI

1) Art → If G, G' be two groups and $f : G \to G'$ is isomorphism prove that
(i) $f(e) = e'$, where e and e' are the identity element of G and G' respectively.
(ii) $f(a^{-1}) = [f(a)]^{-1}$ where a^{-1} is the inverse of $a \in G$

OR

(i) P.T. The identity element G and G' are correspond (or preserved)
(ii) P.T The inverse of any element of G and G' are correspond (preserved)

Proof: Let $(o, *)$ be two group operations of G and G' respectively.
(i) If e and e' are the identity elements of G and G' respectively Then we have to prove.

$$f(e) = e'$$

Let $a' \in G'$ be arbitrary
$\because f$ is isomorphism so there exists only one element say $a \in G$ such that $f(a) = a'$

We have,

$$a \circ e = e \circ a = a \quad \forall \, a \in G$$

~~taking $a \circ e = a$~~
~~$\Rightarrow f(a \circ e) = f(a)$~~
~~$\Rightarrow f(a) * f(e) = f(a)$~~
~~$\Rightarrow a' * f(e) = a'$~~

$\Rightarrow f(a \circ e) = f(e \circ a) = f(a)$
$\Rightarrow f(a) * f(e) = f(e) * f(a) = a'$
$\Rightarrow a' * f(e) = f(e) * a' = a'$

~~Similarly $e \circ a = a$~~

~~$\Rightarrow f(e) * f(a) = f(a)$~~

~~$\Rightarrow f(e) * a' = a'$~~

$\therefore a' * f(e) = f(e) * a' = a' \;\forall a' \in G'$

$\Rightarrow f(e)$ is the identity element of G'. Since identity element of a group is unique so $f(e) = e'$

(ii) Let a be any arbitrary element of G and a^{-1} be inverse of a.

$\therefore a \circ a^{-1} = a^{-1} \circ a = e$

~~taking $a \circ a^{-1} = e$~~

~~$\Rightarrow f(a \circ a^{-1}) = f(e)$~~

~~$\Rightarrow f(a) * f(a^{-1}) = e'$~~

~~again taking,~~

~~$a^{-1} \circ a = e$~~

~~$\Rightarrow f(a^{-1} \circ a) = f(e)$~~

~~$\Rightarrow f(a^{-1}) * f(a) = e'$~~

~~$f(a) * f(a^{-1}) = f(a^{-1}) * f(a) = e'$~~

$\Rightarrow f(a \circ a^{-1}) = f(a^{-1} \circ a) = f(e)$

$\Rightarrow f(a) * f(a^{-1}) = f(a^{-1}) * f(a)$
$= e'$

$\Rightarrow f(a^{-1})$ is the inverse of $f(a)$. since inverse of each of a group is unique so

$f(a^{-1}) = [f(a)]^{-1}$

VVVVI

② Art → If f is a homomorphism of a group G into a group G' then prove that -

① $f(e) = e'$ where e, e' are identity element of G and G' resp.

② $f(a^{-1}) = [f(a)]^{-1} \;\forall a \in G$

⇒ ① ~~Let $a \in G$ then $f(a) \in G'$ and $(\circ, *)$ be the group operation of G and G', we have~~

~~$f(a) * e' = f(a)$~~

③ → Let b' be any arbitrary element of G'. since f is onto $\exists$ an element b in G that $f(b) = b'$ again since G is a cyclic group generated by the element a. Each element of G is expressible as some integral powers of a

Let $b = a^m$

$\Rightarrow f(b) = f(a^m) = f(a \circ a \circ a \ldots \circ a)$
$= f(a) * f(a) * \cdots * f(a)$
$= (f(a))^m$

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...t $a \in G$ and $f(a) \in G'$, where $f(a)$ is image. Let $\cdot, *$ be group operations of ... G'.

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$\therefore f(a) * e' = f(a)$

$\Rightarrow f(a) * e' = f(ae)$

$\Rightarrow f(a) * e' = f(a) * f(e)$ [$\because f$ is homomorphism]

$\Rightarrow e' = f(e)$. [by left cancellation law]

...a be any arbitrary element of G and $a^{-1} \in G$ then

$\because a.a^{-1} = a^{-1}.a = e$ — ①

~~$\Rightarrow f(a^{-1})$ is the inverse of $f(a)$~~

~~inverse of each element of a group is unique.~~

~~So. $f(a^{-1}) = [f(a)]^{-1}$~~ $\because f$ is homomorphism, so

$\Rightarrow f(a.a^{-1}) = f(a) * f(a^{-1})$

$\Rightarrow f(e) = f(a) * f(a^{-1})$ — ②

~~next~~, $\Rightarrow e' = f(a) * f(a^{-1})$

~~$f(a^{-1}.a) = f(a^{-1}) * f(a)$~~ $\Rightarrow f(a^{-1})$ is inverse of $f(a)$

~~$\Rightarrow f(e) = f(a^{-1}) * f(a)$~~ Since inverse is

~~from ② and ③~~ unique, so

~~$f(a) * f(a) = f(a^{-1}) * f(a)$~~

$f(a^{-1}) = [f(a)]^{-1}$

...a cyclic group is cyclic