Wien's Displacement Law :-

Statement :- This law states that at the maximum energy associated with the minimum wavelength which is inversely proportiona to the absolute temperature.

If λ_m be the minimum wavelength at t maximum energy and 'T' be the absolute temperature then according to the law

$$\lambda_m \propto \frac{1}{T}$$

$$\boxed{\lambda_m \cdot T = \text{constant}}$$

Derivation :- To derive Wien's displacement law, l us consider a spherical enclosure capab of expanding radially just like football bladder having reflecting walls. let the sphere is full of diffused radiation of en density 'u' at a uniform temperature T 'V' be the volume of the enclosure. The the total energy of radiation

$$\boxed{U = u \cdot V} \quad \text{—— (1)}$$

Since the enclosure be expanded adiabatically with uniform velocity 'v' then the work done can be written as

$$\boxed{dW = P.dV} \quad \text{——} \quad (2)$$

We have from first law of thermodynamics

$$\boxed{dQ = dU + dW} \quad \text{——} \quad (3)$$

We know that for the adiabatic change

$dQ = 0$ (zero)

and

$dW = PdV$

But for diffused radiation

$$P = \frac{1}{3} \cdot u$$

$$\therefore \; W = \frac{1}{3} u \cdot dV$$

Putting all the values in eq (3)

$$0 = d(u \cdot V) + \frac{1}{3} u \cdot dV$$

$$u \cdot dV + V \cdot du + \frac{1}{3} u \cdot dV = 0$$

$$\frac{4}{3} u \cdot dV + V \cdot du = 0$$

dividing throughout by 'u'

$$\frac{4}{3} \frac{\cancel{u}}{\cancel{u}} \cdot dV + \frac{V}{u} \cdot du = 0$$

$$\frac{4}{3} dV + \frac{V}{u} \cdot \frac{du}{u} = 0$$

Again dividing throughout by 'V'

$$\frac{4}{3} \frac{dV}{V} + \frac{du}{u} = 0$$

Integrating both sides

$$\frac{4}{3}\int\frac{dv}{v} + \int\frac{du}{u} = 0$$

$$\frac{4}{3}\log v + \log u = \log k$$

where 'k' is the integration constant

$$\log v^{4/3} + \log u = \log k$$

$$\log (v^{4/3} \cdot u) = \log k$$

$$\boxed{u \cdot v^{4/3} = k} \text{ (constant)}$$

According to Stefan's law

$$\boxed{u = aT^4} \text{ ——— } (5)$$

where a is a constant quantity putting this value of u in eq (4)

$$v^{4/3} \cdot aT^4 = \text{constant}$$

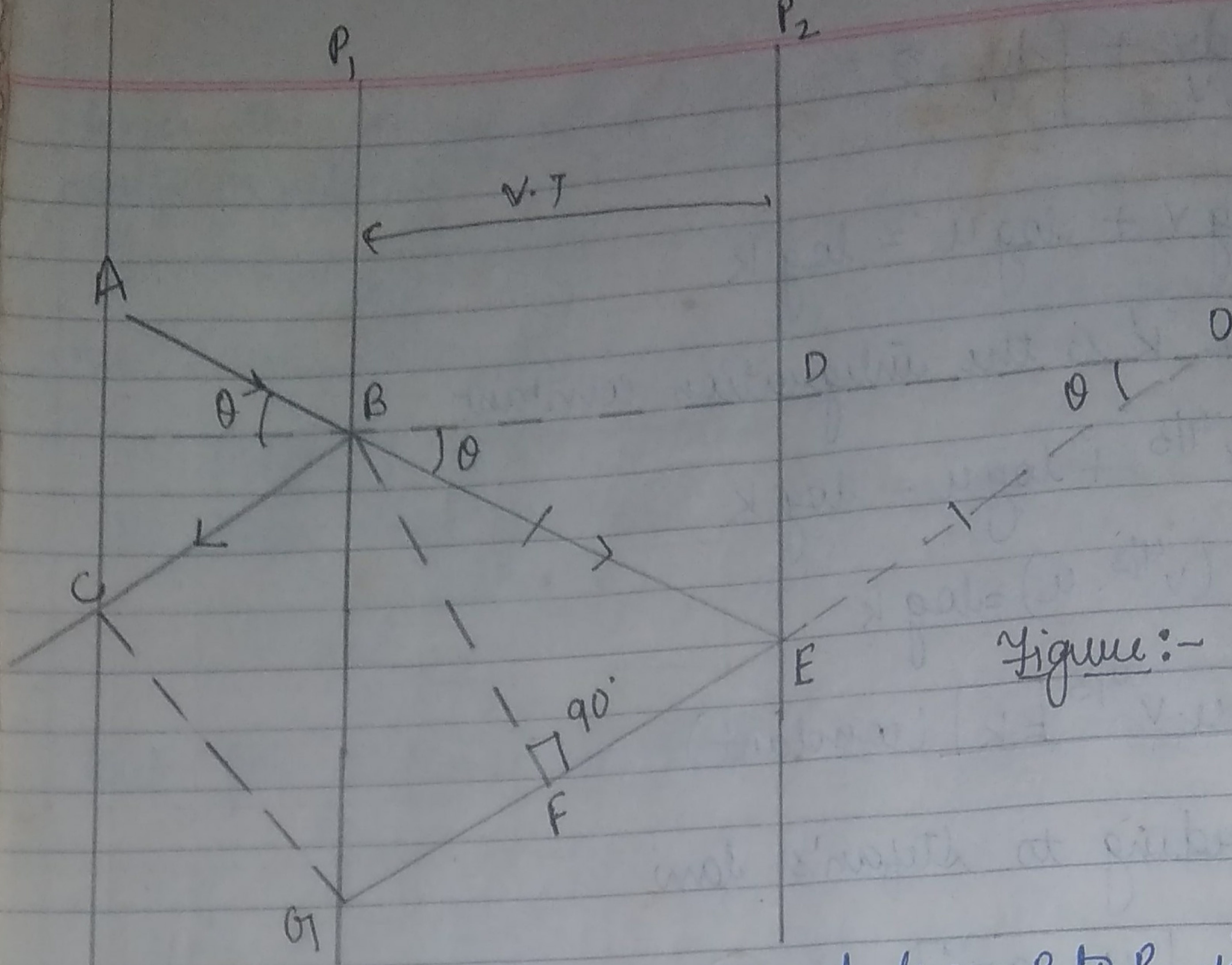
$$v^{4/3} \cdot T^4 = \frac{\text{constant}}{a}$$

$$(v^{1/3} \cdot T)^4 = (\text{constant})^4$$

$$\boxed{v^{1/3} \cdot T = \text{Constant}} \text{ ——— } (6)$$

As we know that the enclosure expand Just like football bladder and hence the reflected ray will have change in wavelength due to doppler effect for this let us consider a ray AB of wavelength 'λ' be made to fall on the wall of the enclosure at an angle 'θ' and Hence the ~~wave~~ reflect toward BC. Therefore the wall expands with the velocity 'v' and attains the new positions by covering a distance as shown below

Figure :-



Therefore, The distance covered from P_1 to P_2 is 'V.T'.
Now the next wave will suffer the reflection from the new position E to EG. If the distance BC = λ, then for the second surface the total distance covered by the wave is BE+EG which is equal to the another wavelength 'λ''.
Hence,

$$BC = \lambda$$

$$BE + EG = \lambda'$$

Now the change in wavelength

$$d\lambda = \lambda' - \lambda$$

$$\boxed{d\lambda = (BE + EG) - BC} \quad \text{——— (7)}$$

We have draw two perpendicular BF and CG hence from the diagram

$$BC = FG$$

Putting this value in eq (7)

$$d\lambda = BE + EG - FG$$

$$= BE + EF + \cancel{FG} - \cancel{FG}$$

$$\boxed{d\lambda = BE + EF} \quad \text{——— (8)}$$

From the diagram

$BE = EO$

Hence eq (8) becomes

$d\lambda = EO + EF$

$$\boxed{d\lambda = OF} \text{ —— } (9)$$

From the diagram In $\triangle$le BOF

$$\cos\theta = \frac{OF}{OB}$$

$$\boxed{OF = OB\cos\theta} \text{ —— } (10)$$

Hence the eq (9) becomes

$d\lambda = OB\cos\theta$

But,

$BD = OD$

$OB = BD + OD$

$= BD + BD$

$= 2BD$

But $BD = V \cdot T$

$$\boxed{OB = 2VT}$$

Putting this value of OB in eq (10)

$$\boxed{d\lambda = 2VT\cos\theta} \text{ —— } (11)$$

If 'c' be the velocity of the wave then we know that

$$c = \frac{\lambda}{T}$$

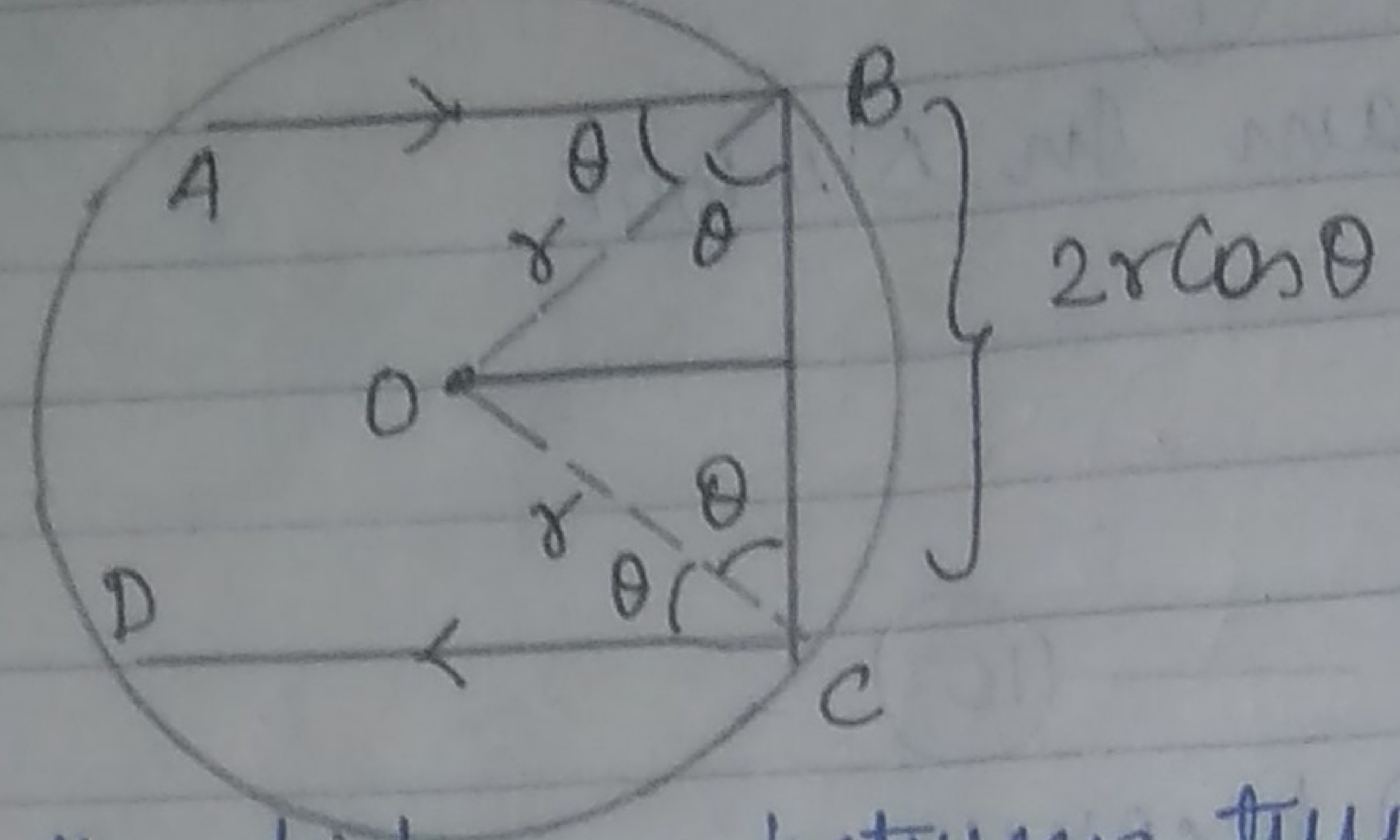
$$\boxed{T = \frac{\lambda}{c}}$$

Putting this value of 'T' in eq (11)

$$\boxed{d\lambda = \frac{2V\lambda}{c}\cos\theta} \text{ —— } (12)$$

i.e.

Now when the radiation is incident inside the spherical enclosure then it suffers multiple reflection and each reflected ray will make an angle 'θ' with the normal as shown below:-



Therefore the distance between two consecutive reflection is $2r\cos\theta$ where r is the radius of the spherical closure

The time taken by the wave between two consecutive reflection

$$= \frac{2r\cos\theta}{c}$$

Therefore, the no. of reflection per second

$$= \frac{c}{2r\cos\theta}$$

Now, the no. of reflection in time dt is equal to

$$= \frac{c}{2r\cos\theta} \cdot dt \quad \text{——— (13)}$$

Due to the diffused radiations the radius of the spherical enclosure changes by 'dr' in time 'dt'

$$V = \frac{dr}{dt}$$

$$dt = \frac{dr}{}$$

Putting this value of dt in eq (13)

No. of reflection in time 'dt' is equal to

$$= \frac{c}{2r\cos\theta} \cdot \frac{dr}{v}$$

Now, the change in wavelength in time dt is equal to

change in wavelength at one reflection × No. of reflection per second × dt

$$d\lambda = \frac{\cancel{c}}{\cancel{2}r\cancel{\cos\theta}} \times \frac{\cancel{2v}\lambda\cancel{\cos\theta}}{\cancel{c}} \times \frac{dr}{\cancel{v}}$$

$$d\lambda = \frac{\lambda\, dr}{r}$$

$$\frac{d\lambda}{\lambda} = \frac{dr}{r}$$

Integrating both the sides:-

$$\log \lambda = \log r + \log k$$

$$\log \lambda - \log r = \log k$$

$$\log \frac{\lambda}{r} = \log k$$

$$\boxed{\frac{\lambda}{r} = k = \text{constant}} \quad \text{——— (14)}$$

We know that volume of the sphere

$$V = \frac{4}{3}\pi r^3$$

Putting this value of V in eq (6)

$$\left(\frac{4}{3}\pi r^3\right)^{1/3} \cdot T = \text{Constant}$$

$$\boxed{r \cdot T = \text{Constant}} \quad \text{——— (15)}$$

From the eq (14)

$$r = \frac{\lambda}{\text{const}}$$

Putting this value of r in eq (15)

$$\frac{\lambda}{\text{constant}} \cdot T = \text{constant}$$

$$\boxed{\lambda T = \text{constant}}$$

which is the required Wein's Displacement law which is derived from adiabatically.