2 Marks
IMP

(4) P.T every homomorphism image of an abelian group is abelian.

Ans: Let $(G', *)$ be the homomorphism image of an abelian group (G, o) and Let f be the homomorphism.

Let $a', b' \in G'$ be arbitrary

Since f is onto, there exist two elements a and b in G such that $f(a) = a'$ and $f(b) = b'$

$$\therefore a' * b' = f(a) * f(b)$$
$$= f(a \circ b) \quad \{\because f \text{ is isomorphism}\}$$
$$= f(b \circ a) \quad \{\because G \text{ is abelian}\}$$
$$= f(b) * f(a)$$
$$= b' * a'$$

Hence G' is also abelian.

NORMAL SUBGROUP (INVARIENT SUBGROUP)

Defination → A subgroup N of a group G is called normal subgroup iff every left coset of N in G is equal to the corresponding right coset. OR

A subgroup N of a group G is said to be normal iff $xnx^{-1} \in N \ \forall x \in G$ and $\forall n \in N$] $\{ \cancel{xN = Nx} \}$

IMP

(1) Theorem → P.T. a sub group N of a group is normal sub group if and only if $xnx^{-1} \in N$ for all $x \in G$ and for all $n \in N$.

OR

Find the necessary and sufficient condition for normal subgroup.

Ans → First we suppose that N is normal subgroup of G and we have to prove $xnx^{-1} \in N \ \forall x \in G$ and $n \in N$

$\because N$ is normal sub-group

$\therefore xN = Nx \ \forall x \in G$

$\Rightarrow xNx^{-1} = Nxx^{-1} \ \forall x \in G$

$\Rightarrow xNx^{-1} = Ne \quad \forall x \in G$

(Where, e is the identity element of G)

$\Rightarrow xNx^{-1} = N \ \forall x \in G$

$\Rightarrow xnx^{-1} \in N \ \forall x \in G$ and $n \in N$

Converse —

Now we shall suppose that $xnx^{-1} \in N \ \forall x \in G$ and $n \in N$ and N is a sub group of G.

To prove that N is normal subgroup of G

$\because xnx^{-1} \in N \quad \forall x \in G$ and $n \in N$

$\Rightarrow xNx^{-1} = N \quad \forall x \in G$

$\Rightarrow xNx^{-1} = Ne \quad \forall x \in G$

$\Rightarrow xNx^{-1} = Nxx^{-1} \quad \forall x \in G$

$\Rightarrow xN = Nx \quad \forall x \in G$ (by right cancellation law)

Hence N is normal sub-group of G.

VVI Theorem (2)

A subgroup H ~~and~~ is a normal sub group of G if and only if each left acoset of H in G is a right coset of H in G.

Ans: Let H be a normal sub-group of G, then $xhx^{-1} \in H \ \forall x \in G$ and $\forall h \in H$

$\Rightarrow \cancel{xh \in Hx \ \forall x \in G \text{ and } h \in H}$

$\Rightarrow \cancel{xh \in Hx \ \forall x \in G}$

$\because nhn^{-1} \in H$

~~Also~~

$\Rightarrow nHn^{-1} = H$

$\cancel{xhx^{-1} \in H \ \forall x \in G \text{ and } h \in H}$

$\Rightarrow nHn^{-1}n = Hn$

$\Rightarrow \cancel{x^{-1}h(x^{-1})^{-1} \in H \ \forall x \in G \text{ and } \forall h \in H}$

$\Rightarrow nHe = Hn$

$\Rightarrow \cancel{x^{-1}hx \in H \ \forall x \in G \text{ and } \forall h \in H}$

$\Rightarrow nH = Hn$

$\therefore$ each left coset is equal to right coset.

Converse,

Let $xH = Hx \quad \forall x \in G$

$\Rightarrow xHx^{-1} = Hxx^{-1} \quad \forall x \in G$

$\Rightarrow xHx^{-1} = He \quad , \forall x \in G$

Where e is the identity element of G

$\Rightarrow xHx^{-1} = H \quad \forall x \in G$

$\Rightarrow xhx^{-1} \in H \quad \forall x \in G \, , \forall h \in H$

$\Rightarrow$ H is a normal sub-group of G.

If N_1 and N_2 are any two normal subgroups of
P.T. $N_1 \cap N_2$ is also a normal sub-group of G.

$\because N_1$ and N_2 are two normal subgroups of G

$\Rightarrow N_1$ and N_2 subgroups of G

At first we shall prove that $N_1 \cap N_2$ is also sub

$\because N_1 \cap N_2 \subseteq N_1 \subseteq G$

$\Rightarrow N_1 \cap N_2 \subseteq G$

Let n_1, n_2 be two elements of $N_1 \cap N_2$

$\because n_1, n_2 \in N_1 \cap N_2$

$\Rightarrow n_1, n_2 \in N_1$ and $n_1, n_2 \in N_2$

$\Rightarrow n_1 n_2^{-1} \in N_1$ and $n_1 n_2^{-1} \in N_2$ ~~By closure~~

$\Rightarrow n_1 n_2^{-1} \in N_1 \cap N_2$ —— (1) $\{ \because N_1$ and
Subgroup

~~Also, $n_2 \in N_1 \cap N_2$~~

~~$\Rightarrow n_2 \in N_1$ and $n_2 \in N_2$~~

~~$\Rightarrow n_2^{-1} \in N_1$ and $n_2^{-1} \in N_2$ (Since N_1 and N_2 are subg~~

~~$\Rightarrow n_2^{-1} \in N_1 \cap N_2$ then by closure property~~

~~$\Rightarrow n_1 n_2^{-1} \in N_1 \cap N_2$ — (2) [$\because n_1 \in N_1 \cap N_2$]~~

~~from (1) and (2) we get~~

$\therefore n_1 n_2 \in N_1 \cap N_2 \Rightarrow n_1 n_2^{-1} \in N_1 \cap N_2$

$\Rightarrow N_1 \cap N_2$ is a subgroup of G

Now, we shall show that $N_1 \cap N_2$ is also normal. ~~If N_1 a… are normal of G~~.

Let a be any element of G and n be any element $N_1 \cap N_2$.

$\because n \in N_1 \cap N_2 \Rightarrow n \in N_1$ and $n \in N_2$

$\Rightarrow ana^{-1} \in N_1$ and $ana^{-1} \in N_2 \ \forall a \in G ; \forall n \in N_1 \cap N$

[Since N_1 and N_2 are normal subgroups]

$\Rightarrow ana^{-1} \in N_1 \cap N_2 \ \forall a \in G$ and $\forall n \in N_1 \cap N_2$

$\therefore N_1 \cap N_2$ is a normal subgroup of G.

2001 (8 Mark) Theorem (4). P.T. intersection of arbitary normal subgroups of a g is normal subgroup.

Ans → Let $N_1, N_2, N_3, \ldots$ be the normal subgroup group G.

To prove

$\bigcap_{i=1}^{\infty} N_i$ is a normal subgroup of G. First we show

$\bigcap_{i=1}^{\infty} N_i$ is a subgroup of G

$\because \bigcap_{i=1}^{\infty} N_i \subseteq N_i$ ~~…~~ $\subseteq G$

$\Rightarrow \bigcap_{i=1}^{\infty} N_i \subseteq G$

$\therefore \bigcap_{i=1}^{\infty} N_i \subseteq G$

Let a and b any two element of $\bigcap_{i=1}^{\infty} N_i$

$\because a, b \in \bigcap_{i=1}^{\infty} N_i$

$\Rightarrow a, b \in N_i \quad \forall i = 1, 2, 3, \ldots$

$\Rightarrow ab^{-1} \in N_i \quad \forall i = 1, 2, 3 \ldots$

[$\because N_i$ is a subgroup of G]

$\Rightarrow ab^{-1} \in \bigcap_{i=1}^{\infty} N_i$

$\Rightarrow \bigcap_{i=1}^{\infty} N_i$ is also a subgroup of G.

Now we shall show that $\bigcap_{i=1}^{\infty} N_i$ is also normal ~~[illegible]~~
~~where $\forall i = 1, 2, 3 \ldots$ are normal of subgroup of G.~~

Let a be any element of G and n be any element of $\bigcap_{i=1}^{\infty} N_i$

$\because n \in \bigcap_{i=1}^{\infty} N_i \Rightarrow n \in N_i \quad \forall i = 1, 2, 3 \ldots$

$\Rightarrow ana^{-1} \in N_i \quad \forall i \quad \forall a \in G \quad \forall n \in N_i$

{Since N_i are normal subgroup $\forall i = 1, 2, \ldots$}

$\Rightarrow ana^{-1} \in \bigcap_{i=1}^{\infty} N_i \quad \forall a \in G \quad \forall n \in \bigcap_{i=1}^{\infty} N_i$

$\Rightarrow \bigcap_{i=1}^{\infty} N_i$ is normal subgroup of G.

Thus intersection of arbitrary normal subgroup of a group is also normal subgroup.

① P.T. N is a normal subgroup of G iff $xNx^{-1} = N, \forall x \in G$

Ans → Let N be a normal subgroup of G