(ii) $ya=b$ has unique solⁿ in G.

Let e be the identity element of group 'G' and a^{-1} be t[he] inverse of a.

$\therefore \quad aa^{-1} = a^{-1}a = e$ —— (1)

$\because \quad ax = b$ —— (2)

$\Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$ (by associative law)

$\Rightarrow ex = a^{-1}b$ {using (1)}

$\Rightarrow x = a^{-1}b$ —— (3)

Uniqueness:- If possible let $x=c$ be another solⁿ of (2)

$\because x=c$ be a solⁿ of (2)

$\therefore ac=b$

$\Rightarrow a^{-1}(ac) = a^{-1}b$

$\Rightarrow (a^{-1}a)c = a^{-1}b$ (by A.L)

$\Rightarrow ec = a^{-1}b$ {By (1)}

$\Rightarrow c = a^{-1}b$.

$\because$ solⁿ (3) is unique.

Now, we shall show that this solⁿ belongs to G.

$\because a \in G \Rightarrow a^{-1} \in G$

$\therefore a^{-1} \in G,\ b \in G \Rightarrow a^{-1}b \in G$ (by closure proper[ty])

Hence the eqⁿ (2) has unique solⁿ $a^{-1}b$ in G.

The given eqⁿ is

$ya = b$

[Sim]ilarly, we can say that the eqⁿ

$ya=b$ has unique solⁿ ba^{-1} in G.

Note:- (i) The inverse of e is e

(ii) $e^2 = e$

$e^3 = e$ and so on.

(iii) $a^2 = a * a$

$a^3 = a * a * a$

$a^n = a * a * a \cdots * a$ (n times)

Q1 If each element of a group is its own inverse then P.T group is abelian.

or,

If G be a group and $a = a^{-1}$ $\forall a \in G$ then P.T G is abelian

Ans- Let a, b be two arbitrary elements of group G

By question

$a = a^{-1}$, $b = b^{-1}$

$\because$ $a, b \in G \Rightarrow ab \in G$ [by closure property]

$\therefore$ $(ab)^{-1} = ab$

$\Rightarrow$ $b^{-1}a^{-1} = ab$

$\Rightarrow$ $ba = ab$

$\Rightarrow$ G is abelian

2. Prove that the set of all cube roots of unity form an abelian group under ordinary multiplication.

Ans- Let $G = \{1, \omega, \omega^2\}$, where $1, \omega, \omega^2$ are three cube roots of unity

To prove G is abelian group under multiplication

The multiplication table of G as under :-

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3=1$
ω^2	ω^2	1	$\omega^4=\omega$

$\{\because \omega^3=1\}$

(i) from the table we see all elements of the table are member of G. Therefore closure property hold in G.

(ii) Since the elements of G are either real or complex, so they must obey associative law.

(iii) from table, we see 1 is the identity element of G.
(iv) from table we see the inverses of $1, \omega, \omega^2$ are $1, \omega^2, \omega$ re and G is ~~inverse table~~ invertiable.
from (i) and (iv) its follows that G is a Group under multiplication.
Since the elements of G are either real or complex so must obey commutative law and hence 'G' is also abe group.

) Prove that the set of 4^{th} roots of unity form ~~a~~ an ~~abelian~~ abeli group under multiplication.
- Let $G=\{1,-1,i,-i\}$, where $1,-1,i,-i$ are 4^{th} roots of unity
To prove G is abelian group under multiplication
The multiplication table 'G' is under:-

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-1	i	1	-1

$\{ a * a^{-1} = e$
$\&$ here $e = 1 \}$

(i) from this table are we see all elementry of the table are the member of G so G is ~~not~~ closed.

(ii) Since the elements of 'G' are either real or complex so they must obey a associative law.

(iii) From table we see 1 is identity element of G

(iv) from table we see the inverse of 1, -1, i, -i are 1, -1, -i, i resp. and so G is ~~inverse table~~. invertiable.

from (i) and (iv) its follows that G is a group under multiplication.

Since the elements of G either real or complex so they must obey the commutative law and hence G is also abelian group under multiplication.

Note – n^{th} roots of unity

$$(1)^{1/n} = (\cos 2\pi r + i \sin 2\pi r)^{1/n}$$

$$= \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}, \quad r = 0, 1, 2, \ldots, n-1$$

$$= e^{i 2r \frac{\pi}{n}}$$

where $r = 0, 1, 2, \ldots, (n-1)$