Problem (18) Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$

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Ans Given equation is written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2$$

Here $P = -\frac{1}{x}$, $Q = -4x^2$, $R = 8x^2\sin x^2$
Changing the independent variables from x to z.
The equation reduces to the form

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R \quad \text{---} (1)$$

where $P_1 = \dfrac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$. Let us choose z as $P_1 = 0$

Then $P\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0 \Rightarrow -\frac{1}{x}\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0$

Putting $\frac{dz}{dx} = v$ Then above Equation is in

the form $-\frac{1}{x}v + \frac{dv}{dx} = 0$

$\Rightarrow \frac{dv}{dx} = \frac{v}{x} \Rightarrow \frac{dv}{v} = \frac{dx}{x}$. Integrating

$\log v = \log x \Rightarrow v = x \Rightarrow \frac{dz}{dx} = x$

$\Rightarrow dz = x\,dx$, Integrating

$\Rightarrow z = \frac{x^2}{2} \quad \text{---} (2)$

Now $Q_1 = \frac{-4x^2}{x^2} = -4$ and $R_1 = \frac{8x^2\sin x^2}{x^2} = 8\sin x^2 = 8\sin 2z$

Hence Eq (1) reduces to the form.

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

$$\Rightarrow \frac{d^2y}{dz^2} + (-4)y = 8\sin 2z$$

$$\Rightarrow \frac{d^2y}{dz^2} - 4y = 8\sin 2z \quad \text{--- (3)}$$

$$\Rightarrow (D^2 - 4)y = 8\sin 2z, \text{ where } D \equiv \frac{d}{dz}$$

Now A.E. is $m^2 - 4 = 0$

$$\Rightarrow m^2 = 4 \Rightarrow m = \pm 2$$

C.F. is $y = C_1 e^{-2z} + C_2 e^{2z}$.

P.I. $y = \frac{8}{D^2-4} \sin 2z$

$$= \frac{8}{-2^2-4} \sin 2z = -\frac{8}{8} \sin 2z$$

$$= -\sin 2z$$

Hence the complete solution is

$$y = C.F. + P.I.$$

$$= C_1 e^{-2z} + C_2 e^{2z} - \sin 2z$$

$$= C_1 e^{-x^2} + C_2 e^{x^2} - \sin x^2 \quad \text{by (2)}$$

Ans