Boolean Algebra

Date | Page No

Defⁿ: A non-empty set B together with two binary operations "+" & "•" (known as addition & multiplication) and an unary operation "'" (known as complementation) on B is said to be Boolean Algebra if it satisfies the following axioms for $a, b, c \in B$.

① (a) $a + b = b + a$
(b) $a \cdot b = b \cdot a$ } → Commutative prop.

② (a) $a + (b \cdot c) = (a+b) \cdot (a+c)$
(b) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ } → Distributive prop

③ $\exists$ 0 & $1 \in B$ such that
(a) $a + 0 = a$
(b) $a \cdot 1 = a$ } Existence of identity

④ For every $a \in B$ $\exists$ $a' \in B$ such that
(a) $a + a' = 1$
(b) $a \cdot a' = 0$ } Existence of complement

A Boolean Agebra is denoted by tuples $(B, +, \cdot, ', 0, 1)$ or by $(B, +, \cdot)$ or by B.

Alternate Defⁿ: → Let B be any set of statements and let $\vee$ (OR, JOIN), $\wedge$ (AND, MEET) and $\sim$ be operations on B. Then the mathematical structure $(B, \vee, \wedge, \sim)$ is said to be a Boolean Algebra if it satisfies the following axioms.

① Commutative Laws.
(a) $a \vee b = b \vee a$
(b) $a \wedge b = b \wedge a$ $\forall\, a, b \in B$

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Date | Page No.

② **Distributive laws.**

(a) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(b) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \forall\, a, b, c \in B$.

③ **Identity laws** : $\exists$ exists $0, 1 \in B$ such that

(a) $a \vee 0 = 0 \vee a = a$

(b) $a \wedge 1 = 1 \wedge a = a \quad \forall\, a \in B$

④ **Complement laws:** For every $a \in B$ $\exists \sim a \in B$ such that (a) $a \vee \sim a = 1$

(b) $a \wedge \sim a = 0$

Example: Let S b an non-empty set and $P(S)$ be the power set of S. Then show that $P(S)$ is a Boolean algebra with respect to union & intersection as two binary operations and complement of a set as unary operation. Considering ϕ and S as 0 and 1 respectively.

Proof Let $A, B \in P(S)$

$\Rightarrow$ A and B are subsets of S.

We know that union & intersection of any two sets is always satisfy commutative laws

ie $A \cup B = B \cup A$

& $A \cap B = B \cap A$

So commutative laws hold.

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Let $A, B, C \in P(S)$

We know that for any three sets A, B & C

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

So distributive laws hold.

Also $\phi, S \in P(S)$ so that for any $A \in P(S)$

we have $A \cup \phi = \phi \cup A = A$

$A \cap S = S \cap A = A$

$\Rightarrow$ ϕ & S are identities for $\cup$ & $\cap$ respectively

Also for $A \in P(S)$ $\exists$ $S - A \in P(S)$ such that

$A \cup (S-A) = S$

& $A \cap (S-A) = \phi$

So for every $A \in P(S)$ we have $(S-A)$ its compliment

i.e $A' = S - A$.

Thus $(P(S), \cup, \cap, ', \phi, S)$ is a Boolean Algebra.

Properties

Thm In a Boolean Algebra

(I) Additive identity is unique

(II) Multiplicative identity is unique.

(III) Complement of every element is unique.

Proof Left for students

Date			
Page No.			

Idempotent laws

If a be an element of a Boolean Algebra then (i) $a+a=a$

(ii) $a \cdot a = a$.

Proof (i) Let $a \in B$ then $a+a \in B$. [$\because$ + is binary operation]

Now

$$a+a = (a+a)\cdot 1 \quad (\because 1 \text{ is multiplicative identity})$$
$$= (a+a)\cdot(a+a') \quad (\text{Complement law})$$
$$= a+(a\cdot a') \quad (\text{Distributive law})$$
$$= a+0 \quad (\text{Complement law})$$
$$= a \quad (\text{Additive identity})$$

(Proved)

(ii) Let $a \in B$ then $a \cdot a \in B$ ($\because \cdot$ is binary operation)

Now

$$a\cdot a = a\cdot a + 0 \quad (\text{Additive identity})$$
$$= a\cdot a + a\cdot a' \quad (\text{Complement law})$$
$$= a\cdot(a+a') \quad (\text{Distributive law})$$
$$= a\cdot 1 \quad (\text{Complement law})$$
$$= a \quad (\text{Multiplicative identity})$$

(Proved)

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