A. Hashmi
K.C.C.
/5/2020

Linear Equation of 2nd order with variable coefficient, solving by Variation of Parameters

The linear Equation of 2nd degree is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

let $y = u(x)$ and $y = v(x)$ be any linearly independent solutions of Eq (1). where $R=0$. Then the C.F. is $y = Au + BV$. where A and B are constants. Since u and v are solutions of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad (2), \text{ then } \left.\begin{array}{l} u_2 + Pu_1 + Qu = 0 \\ \text{and } V_2 + PV_1 + QV = 0 \end{array}\right\} \longrightarrow (3)$$

let A and B are diffble functions of x treating A and B as funs of x. s.t. $y = Au + BV \longrightarrow (4)$ is a solution of Eq (1)

Diffing (4) w.r.t. x two times, find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

Putting $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we have

$A_1u + B_1V = 0$ & $A_1u_1 + B_1V_1 = R$, where A_1 and B_1 contains derivatives.

Solving these two equations by cross-multiplication method, we will get the values of A and B. Putting these values in Eq (4), we will get the complete solution of Eq (1).

Problem (19) solve the equation $\frac{d^2y}{dx^2}+9y=\sec 3x$ by variation of parameters.

Ans Given Equation is $\frac{d^2y}{dx^2}+9y=\sec 3x$

Then the A.E. is $m^2+9=0$

$\Rightarrow m=\pm 3i = 0\pm 3i$

C.F. is $y=e^{0x}(A\cos 3x+B\sin 3x)=A\cos 3x+B\sin 3x$.

Now let A and B are functions of x to be determined

Here $u=\cos 3x$ and $v=\sin 3x$

$u_1=-3\sin 3x$ and $v_1=3\cos 3x$

Then A and B satisfy the equations

$A_1u+B_1v=0$ and $A_1u_1+B_1v_1=R$

$\Rightarrow A_1\cos 3x+B_1\sin 3x=0$

and $A_1(-3\sin 3x)+B_1(3\cos 3x)=\sec 3x$

$$\Rightarrow \left.\begin{aligned} &A_1\cos 3x+B_1\sin 3x+0=0\\ &A_1(-3\sin 3x)+B_1(3\cos 3x)-\sec 3x=0\end{aligned}\right\}$$ solving by C-M method

$$\frac{A_1}{-\sin 3x\sec 3x-0}=\frac{B_1}{0+\cos 3x\sec 3x}=\frac{1}{3\cos^2 3x+3\sin^2 3x}\left(=\frac{1}{3}\right)$$

$$\left.\begin{aligned} A_1&=\frac{-\sin 3x\sec 3x}{3}=-\frac{\tan 3x}{3}\\ \& \; B_1&=\frac{\cos 3x\sec 3x}{3}=\frac{1}{3}\end{aligned}\right\}$$

$\frac{dA}{dx}=-\frac{\tan 3x}{3}$ & $\frac{dB}{dx}=\frac{1}{3}$

$\int dA=-\frac{1}{3}\int\tan 3x\,dx$ & $\int dB=\frac{1}{3}\int dx$

$\Rightarrow A=-\frac{1}{3}\frac{\log(\sec 3x)}{3}+c_1$ & $B=\frac{1}{3}x+c_2$

8/5/2020 SA Hashmi K.C.C.

Hence finally

$$y = A(x)\, u + B(x)\, v$$

$$\Rightarrow y = \left[-\frac{1}{9}\log(\sec 3x) + c_1\right]\cos 3x + \left[\frac{1}{3}x + c_2\right]\sin 3x$$

$$\Rightarrow y = c_1\cos 3x + c_2\sin 3x - \frac{1}{9}\log(\sec 3x).\cos 3x + \frac{1}{3}x\sin 3x$$

Problem 20 Solve by variation of the parameters of the equation $(D^2-1)y = \sin^2 x$.

Sol Given Eq. is $(D^2-1)y = \sin^2 x$.

A.E. $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

C.F. $y = Ae^{x} + Be^{-x}$ ——— (1)

where A and B are funs of x to be determined

Here $u = e^{x}$ and $v = e^{-x}$.

Hence A and B satisfy the equations

$$A_1 u + B_1 v = 0 \text{ and } A_1 u_1 + B_1 v_1 = R$$

$$\Rightarrow A_1 e^{x} + B_1 e^{-x} = 0 \text{ and } A_1 e^{x} + B_1(-e^{-x}) = \sin^2 x$$

$$\Rightarrow \left.\begin{aligned} A_1 e^{x} + B_1 e^{-x} &= 0 \\ A_1 e^{x} - B_1 e^{-x} &= \sin^2 x \end{aligned}\right\}$$

$2A_1 e^{x} = \sin^2 x$ (on adding)

& $2B_1 e^{-x} = -\sin^2 x$ (On subtracting)

$$\Rightarrow A_1 = \frac{1}{2}e^{-x}\sin^2 x \ \& \ B_1 = -\frac{1}{2}e^{x}\sin^2 x$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{4}e^{-x}(2\sin^2 x) = \frac{1}{4}e^{-x}(1-\cos 2x) = \frac{1}{4}\left[e^{-x} - e^{-x}\cos 2x\right]$$

and $$\frac{dB}{dx} = -\frac{1}{4}e^{x}(2\sin^2 x) = -\frac{1}{4}e^{x}(1-\cos 2x) = \frac{1}{4}\left[e^{x} - e^{x}\cos 2x\right].$$

$$\int dA = \frac{1}{4}\left[\int e^{-x}dx - \int e^{-x}\cos 2x\, dx\right]$$

$$A = \frac{1}{4}\left[-e^{-x} - \frac{e^{-x}(-\cos 2x + 2\sin 2x)}{\sqrt{(-1)^2+2^2}}\right] + c_1$$

$$A = -\frac{1}{4}e^{-x} + \frac{1}{4\sqrt{5}}e^{-x}(\cos 2x - 2\sin 2x) + c_1$$

and $$\int dB = \frac{1}{4}\left[\int e^{x}dx - \int e^{x}\cos 2x\, dx\right]$$

$$B = \frac{1}{4}\left[e^{x} - \frac{e^{x}(\cos 2x + 2\sin 2x)}{\sqrt{1^2+2^2}}\right] + c_2$$

$$= \frac{1}{4}\left[e^{x}\right] - \frac{1}{4\sqrt{5}}e^{x}(\cos 2x + 2\sin 2x) + c_2$$

Hence, the complete solution is

$$y = Au + Bv$$

$$= \left[\cancel{-\frac{1}{4}e^{-x}} + \frac{1}{4\sqrt{5}}e^{-x}(\cos 2x - 2\sin 2x) + c_1\right]e^{x}$$

$$+ \left[\cancel{\frac{1}{4}e^{x}} - \frac{1}{4\sqrt{5}}e^{x}(\cos 2x + 2\sin 2x) + c_2\right]e^{-x}$$

$$= \frac{1}{4\sqrt{5}}e^{0}(\cancel{\cos 2x} - 2\sin 2x - \cancel{\cos 2x} - 2\sin 2x) + \frac{1}{4\sqrt{5}}(c_1 e^{x} - c_2 e^{-x})$$

$$= \frac{1}{4\sqrt{5}}(c_1 e^{x} - c_2 e^{-x}) - \frac{1}{\sqrt{5}}\sin 2x$$