

$$\begin{aligned} & \rightarrow xnx^{-1} \in N \quad \forall x \in G \text{ and } \forall n \in N \\ & \text{clearly } xNx^{-1} \subseteq N \quad \text{--- (1)} \quad \forall x \in G \\ & \therefore xNx^{-1} \subseteq N \\ & \rightarrow x^{-1}N(x^{-1})^{-1} \subseteq N \\ & \Rightarrow x^{-1}Nx \subseteq N \quad \text{--- (2)} \end{aligned}$$

Operating  $x$  to the left and  $x^{-1}$  to the right of (2)

$$\begin{aligned} x(x^{-1}Nx)x^{-1} & \subseteq xNx^{-1} \\ eNe & \subseteq xNx^{-1} \\ \therefore N & \subseteq xNx^{-1} \quad \text{--- (3)} \end{aligned}$$

from (1) and (3)

$$xNx^{-1} = N$$

Conversely:

$$\text{Let } xNx^{-1} = N$$

$$\Rightarrow (xNx^{-1})x = Nx \Rightarrow xN = Nx$$

Then obviously  $N$  is normal subgroups.

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(2) Show that every subgroup of an abelian group is normal.

Ans → Let  $G$  be an abelian group and  $N$  be a subgroup of  $G$ .  
Let  $x$  be any element of  $G$  and  $n$  be any element of  $N$ , then

$$\begin{aligned} xnx^{-1} &= xnx^{-1}n \quad [\because G \text{ is an abelian } x^{-1}n = nx^{-1}] \\ &= en = n \in N \end{aligned}$$

$$\text{Thus } x \in G, n \in N \Rightarrow xnx^{-1} \in N$$

Hence  $N$  is normal in  $G$ .

Note → Since every cyclic group is abelian therefore every subgroup of a cyclic group is normal.

B1 p.T. every subgroup of a cyclic group is normal.

and

(3) If  $G$  is a group and  $H$  is a subgroup of index 2 in  $G$ .  
Prove that  $H$  is a normal subgroup of  $G$ .

Ans. Let  $H$  be a subgroup of group  $G$  of index 2. Then the number of distinct right or left cosets of  $H$  in  $G$  is 2. Let  $x$  be any element in  $G$ . If  $x \in H$  then we have

$$xH = H = Hx \Rightarrow H \text{ is normal}$$

If  $x \notin H$  then right coset  $Hx$  is distinct from  $H$  and the left coset  $xH$  is distinct from  $H$ .

But  $H$  is of index 2.

Therefore, the cosets  $H, Hx, xH$  are such that  $G = H \cup Hx = H \cup xH$ . But there is no element common to  $H$  and  $Hx$  and also there is no element common to  $H$  and  $xH$  and also there is no element common to  $Hx$  and  $xH$ .

Therefore, we must have  $Hx = xH$ .

Thus we have  $Hx = xH \forall x \in G$ .

Hence  $H$  is a normal subgroup of  $G$ .

## NORMALIZER OF AN ELEMENT OF A GROUP

Definition  $\rightarrow$  If  $a \in G$ , then  $N(a)$  is the normalizer of 'a' in  $G$  is the set of all those element of  $G$  which commute with  $a$ .  
i.e.  $N(a) = \{x \in G, xa = ax\}$

(1) Theorem  $\rightarrow$  The normalizer  $N(a)$  of  $a \in G$  is a subgroup of  $G$ .

Ans. We have

$$N(a) = \{x \in G, ax = xa\}$$

Let,  $x_1, x_2 \in N(a)$

Then,  $ax_1 = x_1a$  and  $ax_2 = x_2a$

At first we shall show that  $x_2^{-1} \in N(a)$

$$\begin{aligned}
 &\because ax_2 = x_2a \\
 \Rightarrow &x_2^{-1}(ax_2)x_2^{-1} = x_2^{-1}x_2ax_2^{-1} \\
 &= x_2^{-1}a(x_2x_2^{-1}) = (x_2^{-1}x_2)ax_2^{-1} \\
 &= x_2^{-1}ae = eax_2^{-1} \\
 \Rightarrow &x_2^{-1}a = ax_2^{-1} \\
 \therefore &x_2^{-1} \in N(a)
 \end{aligned}$$

Now, we shall show that  $x_1x_2^{-1} \in N(a)$

$$\begin{aligned}
 a(x_1x_2^{-1}) &= a(x_1)x_2^{-1} = (x_1a)x_2^{-1} \\
 &\quad [\because ax_1 = x_1a] \\
 \Rightarrow x_1(ax_2^{-1}) &= x_1(x_2^{-1}a) \quad [\because ax_2^{-1} = x_2^{-1}a] \\
 &= (x_1x_2^{-1})a \\
 \Rightarrow x_1x_2^{-1} &\in N(a)
 \end{aligned}$$

Thus,  $x_1x_2 \in N(a) \Rightarrow x_1x_2^{-1} \in N(a)$

$\therefore N(a)$  is a sub-group of  $G$ .

Note:-

- (i) It should be noted that  $N(a)$  is not necessarily a normal subgroup of  $G$ .
- (ii) If  $G$  be an abelian group and  $a \in G$  then  $xa = ax \forall x \in G$   
 $\therefore N(a) = G$
- (iii) Since  $ex = xe \forall x \in G$  therefore  
 $N(e) = G$

The center of Group (Parabola)

Definition → The set  $Z$  of all self conjugate ( $a = xax^{-1}$ ) elements of a group  $G$  is called the centre of  $G$ .

Symmetrically,

$$Z = \{Z \in G, ZX = XZ, \forall x \in G\}$$



Theorem

(1) The centre of  $Z$  of a group  $G$  is a normal subgroup of  $G$ .

Ans → We have,

$$Z = \{z \in G, zx = xz \quad \forall x \in G\}$$

At first we shall prove that  $Z$  is a subgroup of  $G$ .

Let  $z_1, z_2 \in Z$  then

$$z_1x = xz_1 \text{ and } z_2x = xz_2 \quad \forall x \in G$$

We have,

$$z_2x = xz_2 \quad \forall x \in G$$

$$\Rightarrow z_2^{-1}(z_2x)z_2^{-1} = z_2^{-1}(xz_2)z_2^{-1}$$

$$\Rightarrow (z_2^{-1}z_2)x(z_2^{-1}z_2) = z_2^{-1}x(z_2z_2^{-1})$$

$$\Rightarrow exz_2^{-1} = z_2^{-1}xe$$

$$\Rightarrow xz_2^{-1} = z_2^{-1}x$$

$$\Rightarrow z_2^{-1} \in Z$$

Now,

$$(z_1z_2^{-1})x = z_1(z_2^{-1}x)$$

$$= z_1(xz_2^{-1}) = (z_1x)z_2^{-1} = (xz_1)z_2^{-1} = x(z_1z_2^{-1})$$

$$\Rightarrow z_1z_2^{-1} \in Z$$

$$\text{Thus, } z_1, z_2 \in Z \Rightarrow z_1z_2^{-1} \in Z$$

$\therefore Z$  is a subgroup of  $G$ .

Now we shall show that  $Z$  is a normal subgroup of  $G$ .

Let  $x \in G$  and  $z \in Z$  then

$$xzx^{-1} = (xz)x^{-1} = (zx)x^{-1}$$

$$= zx = z$$

$$\text{Thus, } x \in G, z \in Z \Rightarrow xzx^{-1} \in Z \quad \forall x \in G$$

$\therefore Z$  is a normal subgroup of  $G$ .

## QUOTIENT GROUP

Definition Let  $N$  be a normal subgroup of Group  $G$  the set of all cosets  $Nx, x \in G$  is denoted by  $G/N$

$$\text{i.e. } \frac{G}{N} = \{Nx : x \in G\}$$

This is actually a group and called quotient group.

(1) Theorem → If  $G$  is a group,  $N$  is normal subgroup of  $G$  then show that  $\frac{G}{N}$  is also a group. OR

The set of all cosets of a normal subgroup is a group with respect to multiplication of composition.

Ans Given  $N$  is normal subgroup of a group  $G$ . Then  $\frac{G}{N} = \{Nx : x \in G\}$  i.e.  $\frac{G}{N}$  is the collection of all cosets of  $N$  in  $G$ , we verify the following group postulates.

(i) Closure property → Let  $x, y \in G$  then

$$x = Nx \text{ and } y = Ny \text{ where } x, y \in G$$

$$\begin{aligned} \therefore (Nx)(Ny) &= N(Nx)y \\ &= N(Nx)y \quad [\because N \text{ is normal, } \therefore Nx = xN] \\ &= N.Nxy = Nxy \\ &= Nz, \text{ where } z = xy \in G \end{aligned}$$

$$\left\{ \begin{array}{l} \text{Since } x, y \in G \Rightarrow xy \in G \\ \therefore Nx, Ny \in \frac{G}{N} \end{array} \right\}$$

$\therefore Nx, Ny \in \frac{G}{N}$ . Hence closure property hold in  $\frac{G}{N}$ .

Thus  $\frac{G}{N}$  is closed with respect to coset multiplication.

Associative Law Let  $a, b, c \in G$  then  $Na, Nb, Nc \in \frac{G}{N}$

Now.

$$\{(Na)(Nb)\}Nc = (Nab)Nc = Nabc$$

and,

$$\begin{aligned} Na\{(Nb)(Nc)\} &= Na(Nbc) \\ &= Na(bc) \end{aligned}$$

But  $(ab)c = a(bc)$  holds in  $G$

Hence  $N(ab)c = N(a(bc))$

$$\Rightarrow \{N(a)(Nb)\}Nc = N(a)\{N(b)(Nc)\}$$

Hence associative law is law in  $\frac{G}{N}$

(iii) Existence of identity  $\rightarrow$  Clearly,  $N = Ne$  is the identity element in  $\frac{G}{N}$ , where  $e$  is the identity element of  $G$ . Since

$$\begin{aligned}(Nx)N &= (Nx)Ne = Nxe = Nx \\ &= (Ne)Nx = N(Nx)\end{aligned}$$

(iv) Existence of inverse  $\rightarrow$  Clearly  $Nx^{-1}$  is the inverse of coset  $Nx$  in  $\frac{G}{N}$  for

$$(Nx)(Nx^{-1}) = Nxx^{-1} = Ne = N$$

Thus each element of  $\frac{G}{N}$  possesses inverse. Hence  $\frac{G}{N}$  is a group with respect to the product of coset.

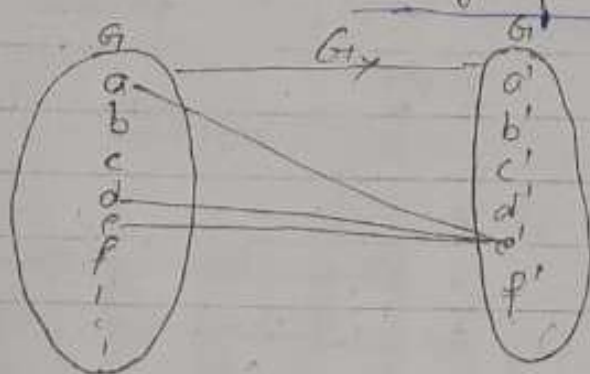
Note  $\rightarrow$  (i) If  $G$  is abelian then  $\frac{G}{N}$  is also abelian for  
 $N(a)(Nb) = N(ab) = N(ba) = (Nb)(Na)$

[ $\therefore G$  is abelian  $\Rightarrow ab = ba$ ]

(ii) If  $G$  is cyclic then  $\frac{G}{N}$  is also cyclic for

$$\begin{aligned}Nb &= Na^n = Na \underbrace{Na \dots Na}_n \text{ times} \\ &= (Na)^n \text{ where } b = a^n\end{aligned}$$

Where  $a$  is generator of  $G$ .



$\therefore Na$  is generator of  $G/N$ .