VVI

Kernal of homomorphism → If (G, o) and $(G', *)$ be two groups and $f: G \to G'$ be homomorphism then kernal of f is the set of those elements of G which mapped to e', identity element of G'

$\therefore K = \ker f = \{x \in G : f(x) = e'\}$

(1999, 2006) VVI If f is homomorphisms of a group G to G', prove that the kernel of the homomorphism f as normal subgroup of G.

OR

P.T. the kernel of homomorphism is an normal subgroup (Invarient subgroup).

Ans> Let f be a homomorphism of a group (G, o) into a group $(G', *)$. Let e and e' be the identity elements of G and G' respectively.

Let K be the kernel of the homomorphism f, so that

$$K = \{x \in G : f(x) = e'\}$$

Since $f(e) = e'$, So K is nonempty

Let $a, b \in K$, Then $f(a) = e'$ and $f(b) = e'$ ~~[∵ f is homomorphism]~~

~~So $f(aob^{-1}) = f(a) * [f(b)]^{-1}$~~ ~~$f(b^{-1}) = [f(b)]^{-1}$~~

$$\text{So, } f(aob^{-1}) = f(a) * f(b^{-1}) \quad [\because f \text{ is homomorphism}]$$
$$= f(a) * [f(b)]^{-1} \quad [\because f(b^{-1}) = [f(b)]^{-1}]$$
$$= e' * (e')^{-1} = e' * e' = e'$$

This shows that $a, b \in K \Rightarrow aob^{-1} \in K$

Hence K is subgroup of G. To show K is normal

Let $x \in G$ and $b \in K$ be arbitary then

$$f(b) = e'$$

$$\therefore f(xobox^{-1}) = f(x) * f(b) * f(x^{-1})$$

$\therefore f(x \circ b \circ x^{-1}) = f(x) * e' * [f(x)]^{-1}$
$= f(x) * [f(x)]^{-1} = e'$

So, $x \circ b \circ x^{-1} \in K \ \forall x \in G$ and $b \in K$

Hence K is a normal subgroup of G.

VVVI Theorem → The necessary and sufficient condition for a homomorphism f of a group G into a group G' with Kernel K to be an Isomorphism of G into G' if that $K = \{e\}$

Proof → Let f be a homomorphism of a group G into a group G'. Let e and e' be the identity elements of G and G' resp. Let K denote the kernel of f. First we suppose that f is an Isomorphism of G into G'

Thus f is one-one

To prove $K = \{e\}$

Let $a \in K$, then $f(a) = e'$ {by defination of kernel}

$f(a) = f(e)$ $\{\because f(e) = e'\}$

$\Rightarrow a = e$ $[\because f$ is one-one$]$

Thus $a \in K \Rightarrow a = e$, So, e is the only element of G which belongs ~~[illegible]~~ to K

$\therefore K = \{e\}$

Converse :- Let $K = \{e\}$

To prove f is an Isomorphism of G into G'

~~i.e. to prove that f is an isomorphism and one-one~~

$\because$ Since f is homomorphism it is given, so far ~~homomorphism~~ to show only f is one-one mapping.

If $a, b \in G$ then

$f(a) = f(b) \Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$\Rightarrow f(a) f(b^{-1}) = f(b) f(b^{-1})$

$\Rightarrow f(ab^{-1}) = f(bb^{-1})$

$\Rightarrow f(ab^{-1}) = f(e)$

$\Rightarrow f(ab^{-1}) = e'$ [$\because f$ is homomorphism]

$\Rightarrow ab^{-1} \in K$

$\Rightarrow ab^{-1} = e$ ($\because K = \{e\}$)

$\Rightarrow ab^{-1}b = eb \Rightarrow ae = eb \Rightarrow a = b$

$\therefore f$ is one-one. Hence f is Isomorphism of G into G'

—x—

<u>Note</u> → The group G and G' are said to be Isomorphisms if there is an Isomorphism between two groups. It is written as

$$G \approxeq G' \text{ or } G \cong G'$$

$\approxeq$ or $\cong$ is called wiggle

<u>VVI</u>

Q → Let f be a homomorphism of the group G into group G' then show that kernel f is a subgroup of G and image f is a subgroup of G'.

<u>Ans</u> → Let f be a homomorphism of a group $(G, \circ)$ into a group $(G', *)$ Let e and e' be the identity elements of G and G' resp. Let K be the kernal of the homomorphism f so that

$$K = \{x \in G ; f(x) = e'\}$$

Since, $f(e) = e'$. So k is non empty. Let a, b be two elements of k.

Then $f(a) = e'$ and $f(b) = e'$

So $f(a \circ b^{-1}) = f(a) * f(b^{-1})$ [$\because f$ is homomorphism]

$= f(a) * [f(b)]^{-1}$ $\quad \because f(b^{-1}) = [f(b)]^{-1}$

$= e' * (e')^{-1} = e'$

This shows that $a, b \in K$

$\Rightarrow ab^{-1} \in k$ Hence k is a subgroup of G.

2nd Part

Let $f(G)$ be the homomorphism image of G.

We have,

$f(G) = \{ f(x) : x \in G \}$

Obviously $f(G) \subseteq G'$

Let, $a', b' \in f(G)$ be arbitrary then, $f(a) = a'$ and $f(b) = b'$ for some $a, b \in G$

Now.

$a'(b')^{-1} = f(a)[f(b)]^{-1} = f(a) f(b^{-1})$ [∵ f is homomorphism]

$= f(ab^{-1})$

Since,

$ab^{-1} \in G \Rightarrow f(ab^{-1}) \in f(G)$

Thus. $a', b' \in f(G) \Rightarrow a'(b')^{-1} \in f(G)$

∴ $f(G)$ is a subgroup of G'

007 Fundamental Theorem on homomorphism (Ist theorem of Isomorphism) of a group.

Statement → If $f : G \to G'$ is homomorphism with Kernal K then $\frac{G}{K} \cong G'$

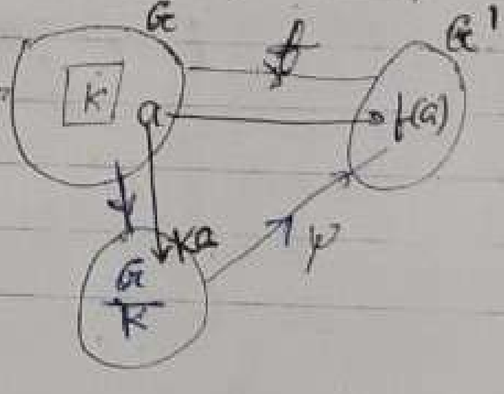
Proof —

Let G' be the Isomorphic image of G and 'f' be the corresponding homomophism. Then f is homomoophism of G onto G'.

Let K be the kernel of this homomorphism thus K is a normal subgroup of G.

We shall prove that

$\frac{G}{K} \cong G'$



In order to prove this we must produce a maping ψ from $\frac{G}{K}$ into G' such that ψ is an Isomorphism.

If $a \in G$, then $Ka \in G/_K$ and $f(a) \in G'$
we define $\psi : \frac{G}{K} \to G'$ such that
$$\psi(Ka) = f(a) \cdot \forall a \in G \quad \text{———} \quad (1)$$
First we shall prove that the mapping ψ is well defined i.e. if $a, b \in G$ and $Ka = Kb$ then $\psi(Ka) = \psi(Kb)$

We have
$$Ka = Kb \Rightarrow Kab^{-1} = Kbb^{-1} = Ke$$
$$\Rightarrow Kab^{-1} = K$$
$$\Rightarrow ab^{-1} \in K$$
$\Rightarrow f(ab^{-1}) = e'$ [where e' is the identity element of G']
$\Rightarrow f(a)\, f(b^{-1}) = e'$ [$\because f$ is homomorphism]
$\Rightarrow f(a)\,[f(b)]^{-1} = e'$ [$\because f(b^{-1}) = [f(b)]^{-1}$
$\Rightarrow f(a)\,[f(b)]^{-1} f(b) = e' f(b)$
$\Rightarrow f(a)\, e' = f(b)$
$\Rightarrow f(a) = f(b)$
$\Rightarrow \psi(Ka) = \psi(Kb)$ {By (1)}
$\therefore \psi$ is well defined.

<u>ψ is one-one</u> :
We have,
$$\psi(Ka) = \psi(Kb)$$
$\Rightarrow f(a) = f(b)$
$\Rightarrow f(a)\,[f(b)]^{-1} = f(b)\,[f(b)]^{-1}$
$\Rightarrow f(a) \cdot f(b^{-1}) = e'$
$\Rightarrow f(ab^{-1}) = e'$ [$\because f$ is homomorphism]
$\Rightarrow ab^{-1} \in K$ [$\because K$ is kernel of f]

$\Rightarrow Kab^{-1} = K$

$\Rightarrow Ka = Kb$

$\therefore \psi$ is one-one

ψ is onto :-

Let a' be any element of G' then $a' = f(a)$ for some $a \in G$ because f is onto

For $Ka \in G/K$ we have

$$\psi(Ka) = f(a) = a'$$

$\therefore \psi$ is onto

Finally, we have $\psi[(Ka)(Kb)] = \psi[Kab]$ [$\because K$ is normal]

$$= f(ab)$$
$$= f(a) f(b)$$
$$= \psi(Ka) \psi(Kb)$$

$\therefore \psi$ is an Isomorphism of $\frac{G}{K}$ onto G'.

Hence $\frac{G}{K} \simeq G'$

Automorphism —

Example → Show that $a \to a^{-1}$ is an automorphism of a group G iff G is abelian.

Ans → Let $f: G \to G$ be automorphism such that

$f(x) = x^{-1} \quad \forall x \in G$

If part: Let G be abelian.

The function f is one-one because.

$f(x) = f(y) \Rightarrow x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1}$

$\Rightarrow x = y$

Also if $x \in G$ then $x^{-1} \in G$ and we have

$f(x^{-1}) = (x^{-1})^{-1} = x \in G$

$\therefore f$ is onto