

Cancellation laws

Thm In a Boolean Algebra B

- (i) if $a+c=b+c$ & $a+c'=b+c'$ then $a=b$
- (ii) if $a \cdot c=b \cdot c$ and $a \cdot c'=b \cdot c'$ then $a=b$

Proof (i) Let $a+c=b+c$ & $a+c'=b+c'$

$$\begin{aligned}
 \text{Now } a &= a+0 && \text{Identity law} \\
 &= a+(c \cdot c') && \text{Complement law} \\
 &= (a+c) \cdot (a+c') && \text{Distributive law} \\
 &= (b+c) \cdot (b+c') && \text{(Given)} \\
 &= b+(c \cdot c') && \text{Distributive law} \\
 &= b+0 && \text{Complement law} \\
 &= b && \text{Identity law}
 \end{aligned}$$

(ii) Let $a \cdot c=b \cdot c$ & $a \cdot c'=b \cdot c'$

$$\begin{aligned}
 \text{Now } a &= a \cdot 1 && \text{Identity law} \\
 &= a \cdot (c+c') && \text{Complement law} \\
 &= a \cdot c + a \cdot c' && \text{Distributive law} \\
 &= b \cdot c + b \cdot c' && \text{Given} \\
 &= b \cdot (c+c') && \text{Complement law} \\
 &= b \cdot 1 && \text{Distributive law} \\
 &= b && \text{Identity law}
 \end{aligned}$$

Boolean Sub-algebra

Let $(B, +, \cdot, ')$ be a Boolean algebra. Let S be any non-empty subset of B . Then S is called Boolean subalgebra of B if S is closed under addition, multiplication & complementation.

IE A subset S of a Boolean algebra B is called a Boolean subalgebra if $a, b \in S$

$$\Rightarrow \begin{cases} a \cdot b \in S \\ a + b \in S \\ a' \in S \end{cases}$$

Note:

1 & 0 always belong to any sub algebra

Th^m A non-empty subset S of a Boolean algebra B is a sub-algebra if it is closed under addition & complementation.

Proof Since a subset of Boolean algebra is sub-algebra if it is closed under addition multiplication & complementation, so it is sufficient to show that S is closed under multiplication.

$$\text{Let } a, b \in S$$

$$\Rightarrow a', b' \in S$$

$$\Rightarrow a' + b' \in S$$

$$\Rightarrow (a \cdot b)' \in S$$

$$\Rightarrow a \cdot b \in S$$

Hence S is closed under multiplication
Proved.

Thm A non-empty subset S of a Boolean algebra B is a sub-algebra if it is closed under multiplication and complementation.

Proof SAME AS PREVIOUS THEOREM

Thm Prove that intersection of two sub-algebra of a Boolean algebra B is also a sub-algebra.

Proof Let B be a Boolean algebra and S_1 & S_2 be subalgebra of B .

$$\text{Let } S = S_1 \cap S_2$$

We want to show that S is also a sub-algebra

Since S_1 & S_2 are subsets of B

$$\Rightarrow S_1 \cap S_2 \text{ is subset of } B$$

$$\Rightarrow S \text{ is subset of } B.$$

$$\text{Now } 0, 1 \in S_1 \text{ \& } 0, 1 \in S_2$$

$$\Rightarrow 0, 1 \in S_1 \cap S_2$$

$$\Rightarrow 0, 1 \in S$$

So S is non-empty.

$$\text{Let } a, b \in S$$

$$\Rightarrow a, b \in S_1 \cap S_2$$

$$\Rightarrow a, b \in S_1 \text{ \& } a, b \in S_2$$

$\Rightarrow a+b, a \cdot b \text{ \& } a' \in S,$
as well as $a+b, a \cdot b \text{ \& } a' \in S,$

$\Rightarrow a+b, a \cdot b \text{ \& } a' \in S_1 \cap S_2$

$\Rightarrow a+b, a \cdot b \text{ \& } a' \in S$

Hence S is sub algebra.

Proposition

A proposition in a Boolean algebra is either a statement or algebraic identity.

Dual of a proposition

If A is any proposition in a Boolean algebra, then a proposition obtained from A by interchanging $+$ and \cdot and exchanging 0 and 1 .

Th^m (Duality Principle).

If a proposition A is derivable from the axioms of a Boolean algebra then its dual is also derivable from those axioms.

Proof

It can easily be verified from all the previous theorems, however proof is not in syllabus.